STABLE RANK ONE IN NONNUCLEAR CROSSED PRODUCTS

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ABSTRACT. We initiate an investigation into the local structure of simple nonnuclear C*-crossed products by showing that stable rank one is generic within two natural classes of minimal actions of free groups on the Cantor set. The arguments also apply to some other free product groups. Our approach is inspired by Li and Niu's stable rank one theorem in the amenable setting and also yields a streamlined argument in that case, along with a generalization to product actions.

CONTENTS

1. INTRODUCTION

Amenability has long established itself as a fundamental tool for probing the finer internal structure of noncommutative operator algebras. The most definitive and consequential classification results in the realms of von Neumann algebras and simple C*-algebras are premised on amenability as a basic hypothesis. This is explained by the fact that, in parallel with Rokhlin-type properties in ergodic theory that permit one to decompose a dynamical system into almost invariant towers, operator-algebraic amenability is associated with robust approximation by finite-dimensional algebras, which provide a versatile combinatorial apparatus for detecting, manipulating, and managing structural data. One can thus view amenability both as a local organizing principle (form) and as an object of study in its own right by way of the algebras that possess it as a property (content).

Date: April 2, 2026.

What is remarkable is that the content doesn't simply reduce to form in the sense of there being a unique amenable object: there are in fact rich classes of amenable von Neumann algebras and C*-algebras, their existence owing to the fact that local matricial structure can be pieced together asymptotically in a great variety of incommensurable ways. There are, to be sure, significant technical and qualitative differences between von Neumann algebras and C*-algebras in the way that this bounty manifests itself, and the very first classification result in the field—the uniqueness of the hyperfinite II_1 factor established by Murray and von Neumann—did reveal there to be a unique separable amenable object among noncommutative probability spaces, i.e., factors possessing a normal tracial state (the abstract notion of amenability for von Neumann algebras actually came later, in various guises including most prominently injectivity, but these were shown by Connes to all be equivalent to hyperfiniteness in the separable case). It was eventually discovered, however, that there is an abundance of nontracial amenable factors and of simple nuclear (i.e., amenable) C*-algebras, both tracial and nontracial.

Amenability can also be applied to great effect even when the algebras in question are themselves either nonamenable or a priori not known to possess good finite-dimensional approximation, as in Connes's use of the McDuff property to prove that injectivity implies hyperfiniteness for separable factors [15] and in Ozawa's proof of his solidity result for the von Neumann algebras of hyperbolic groups, which combines both von Neumann algebraic and C*-techniques [64]. On the more purely C*-algebraic side, amenability is everywhere manifest in the scenography around exactness, nuclear embeddability, and related properties [11], and the questions of simplicity and unique trace for reduced group C*-algebras were discovered in [10, 43] to hinge precisely on the threshold between amenability and nonamenability within the group.

In the present work we take this principle in a new direction by using amenability—in the form of Rokhlin tower decompositions and Følner tilings—to show that nonnuclear tracial reduced crossed products coming from minimal actions of free groups on the Cantor set frequently have stable rank one. Inspired by the relation of Bass stable rank to cancellation phenomena in algebraic K-theory, Rieffel introduced the notion of (topological) stable rank as a dimension-type invariant that is similarly related to cancellation issues in (topological) K-theory within the framework of unital C*-algebras [68]. The property of stable rank one boils down to the density of invertible elements and implies both that the Murray–von Neumann semigroup has cancellation and that K_1 can be expressed without stabilizing as the unitary group modulo the path component of the identity. It is the operative hypothesis in the analysis of divisibility phenomena that has driven recent progress on the Cuntz semigroup and its applications to longstanding problems like rank realization, which Thiel succeeded in confirming for all simple C*-algebras of stable rank one [69, 79, 4]. Many simple separable unital finite C*-algebras are known to have stable rank one, notably those satisfying the regularity property of $\mathcal{Z}$-stability that has come to play a key role in the Elliott classification program [73], and more generally those that are pure in the sense of Winter (i.e., have strict comparison and are almost divisible) [55]. Stable rank one also holds for crossed products of free minimal actions of FC (and in particular Abelian) groups on compact metrizable spaces [1, 53, 57, 59], some of which fail to be $\mathcal{Z}$-stable even when the group is $\mathbb{Z}$ [29], as well as for the reduced group C*-algebras of free groups and, more generally, acylindrically hyperbolic groups [23, 22, 28, 67]. On the other hand, Villadsen showed in [81] that all possible values of stable rank are realized by simple separable nuclear C*-algebras. Moreover, the stable rank of a simple C*-algebra is infinite as soon as a nonunitary isometry is present. Granted that one stays within the realm of finite C*-algebras, however, one can interpret stable rank one as a regularity condition expressing a kind of zero-/one-dimensionality. We direct the reader to [75] for a state-of-the-art picture of how stable rank one interweaves into the structure and classification theory of nuclear C*-algebras.

Despite all of this progress, little seems to be known about the value of stable rank for naturally arising nonnuclear simple separable finite C*-algebras, in particular those arising as crossed products, once one moves beyond the reduced group C*-algebras treated in [23, 22, 28, 67, 3, 80, 65]. The arguments in these papers rely on spectral or topological-dynamical phenomena tied to the geometry of the group, and it is not clear whether they can be adapted to handle crossed products. Our approach, being rooted in amenability of a type II_1 nature (i.e., Følnerness and invariant measures), is completely different. As we are working in the Baire category framework, we also need to develop some of the descriptive set theory of spaces of actions of free groups and other free products on the Cantor set, which does not seem to have been explored much in the literature before the recent appearance of the paper [20].

We now formulate our main results. For a countable discrete group G and a compact metrizable space X, we write $\mathrm{WA}(G,X)$ for the set of all topologically free minimal actions $G \curvearrowright X$ that are weakly mixing and admit an invariant Borel probability measure. We equip this with the topology of elementwise compact-open convergence, which is Polish (see Sections 2 and 8 for more details).

Theorem A. *Let G be a residually finite countable discrete group and H an amenable countable discrete group containing a normal infinite cyclic subgroup. Let X be the Cantor set. For a generic action in $\mathrm{WA}(G*H,X)$ the reduced crossed product $C(X)\rtimes_\lambda (G*H)$ has stable rank one.*

Given that the free group F_d on $d \geq 2$ generators can be written as the free product $F_{d-1} * \mathbb{Z}$, we obtain the following as a special case. The crossed products below, and more generally all tracial reduced crossed products arising from minimal actions of free groups on compact metrizable spaces, are known to be MF algebras [48].

Corollary B. *Let $d \geq 2$ and let X be the Cantor set. For a generic action in $\mathrm{WA}(F_d,X)$ the reduced crossed product $C(X)\rtimes_\lambda F_d$ has stable rank one.*

To establish Theorem A we introduce a dynamical notion of *square divisibility* that distills some of the operator-algebraic structure used by Li and Niu in [53] to prove a stable rank one result in the context of amenable acting groups. Like them, we follow the basic strategy pioneered by Rørdam in [71] in which elements a satisfying $ad = da = 0$ for some nonzero positive element d are unitarily rotated into nilpotent elements (see the introduction to Section 4), a procedure that square divisibility permits us to implement in our crossed product context. In fact we develop two versions of this square divisibility (Definitions 3.3 and 3.10) which are tailored to two different applications although they both imply stable rank one, as we show in Theorems 4.6 and 4.7 by applying some of the techniques from [53]. The second version, called *weak square divisibility*, is only defined over the Cantor set but has the additional virtue that it is a G_δ property by the way the definition is locally formulated in terms of open conditions, so that we can apply the Baire category theorem in a natural manner. After a series of lemmas that serve to establish the density of the open sets at play in the definition of weak square divisibility, we are thereby able to deduce, under the hypotheses of Theorem A, that the weakly squarely divisible actions in $\mathrm{WA}(G*H,X)$ form a dense G_δ set (Theorem 9.2). In conjunction with Theorem 4.7, this yields Theorem A.

In order to establish the aforementioned density we also need to invoke the stronger form of square divisibility for minimal actions of countably infinite amenable groups satisfying (dynamical) comparison and the uniform Rokhlin property (URP), a fact that we establish in Theorem 5.7 in the general setting of compact metrizable spaces. For the purposes of Theorem A we only need to worry about the Cantor set and do not need to know that square divisibility itself implies stable rank one. We show nevertheless in Theorem 4.6 that the latter implication does indeed hold under more general compact metrizable hypotheses, and note in Theorem 4.7 that essentially the same argument also gives the implication for weak square divisibility in the Cantor setting, which is what we use to derive Theorem A. Theorems 4.6 and 5.7 in combination show that, for minimal actions of countably infinite amenable groups on compact metrizable spaces, the URP and comparison together imply that the crossed product has stable rank one, so that we recover the stable rank one result of Li and Niu from [53] in a weaker form that replaces their hypothesis of Cuntz comparison on open sets (COS) with the purely dynamical notion of comparison (actually neither of the latter two properties is known to fail among minimal actions, and comparison holds in all of the cases where COS is known to hold). We thus obtain, for example, a streamlined proof of stable rank one for the crossed products of free minimal actions of infinite Abelian groups on compact metrizable spaces, with comparison in this case having been proved in [57] and the URP in [59]. Our methods also allow for the following generalization to product actions, which is a consequence of Theorems 4.6 and 6.1.

Theorem C. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be minimal actions of countable discrete groups on compact metrizable spaces with G infinite, and suppose that the first action has the URP and comparison. Then the reduced crossed product of the product action $G \times H \curvearrowright X \times Y$ given by $(g,h)(x,y) = (gx, hy)$ has stable rank one.*

That the conjunction of the URP and comparison should really be considered a single property, in analogy with almost finiteness as the combination of almost finiteness in measure and comparison, has been borne out in recent work of Naryshkin in [59], which establishes several equivalent formulations of this conjunction, terminologically abbreviated to URPC, and derives some remarkable applications to shift embeddability.

Our approach to stable rank one in the amenable setting, which underpins not only Theorem C but also Theorem A (as well as Theorem D below), differs from that of Li and Niu by leveraging the Rokhlin towers coming from the URP to greater effect so as to create, as in the definition of square divisibility, an array of open sets whose complement is small but whose boundaries are in turn much smaller than this complement. It is this relative smallness at two different scales that allows us to work with a simpler version of the apparatus at play in [53].

The proof of Theorem C goes in the direction of trying to show that if $G \curvearrowright X$ is squarely divisible then so is every product action of the form $G \times H \curvearrowright X \times Y$ (a stronger statement that we have been unable to verify), but we need some extra local information beyond square divisibility that we get from the URP and comparison. It is nevertheless interesting to compare Theorem C with recent results on the permanence of dynamical and C*-algebraic regularity properties under taking products. Among regularity properties for C*-algebras, $\mathcal{Z}$-stability hits the sweet spot of being both highly consequential (classification being the highest payoff) and accessible to verification in a great many cases. An illustration of its robustness is the trivial fact that a minimal tensor product $A \otimes_{\min} B$ is $\mathcal{Z}$-stable as soon as one of the factors is. In particular, the reduced crossed product of a product action $G \times H \curvearrowright X \times Y$ is $\mathcal{Z}$-stable as soon as this is the case for one of the factors. Once one replaces $\mathcal{Z}$-stability by a weaker regularity property like stable rank one, however, the issue of permanence under taking product actions can become quite tricky. This is even already true for almost finiteness, the closest dynamical analogue to $\mathcal{Z}$-stability that we have in the setting of amenable acting groups. See [52], where the permanence under products of both almost finiteness and almost finiteness in measure was established using C*-algebra technology, and also [46], where it was shown for the related small boundary property using purely dynamical methods.

Theorem C is not as unrelated to Theorem A as it might appear at first glance. The proof of Theorem A also leverages the URP and comparison, via Theorem 5.7, in the context of a product construction. In that case however we are approximating a given action of $G * H$ via its diagonal product with another action, and so we do not have the freedom of starting from separate actions of two different groups as in Theorem C. This renders the analysis much more complicated, although the verification of square divisibility is similar in the two settings. In particular, for Theorem A we need to build a machine for producing diagonal actions that are weakly mixing and minimal, a task that involves some ergodic theory and is carried out in Section 7.

Unfortunately we have been unable to show that the generic action in Theorem A or Corollary B does not belong to a single conjugacy class, although we strongly suspect that every conjugacy class is meagre, and indeed this is known to be the case among weakly mixing minimal actions of $\mathbb{Z}$ on the Cantor set [40, Theorem 1.2 and Lemma 8.1]. On the other hand, a generic minimal action of F_d on the Cantor set admitting an invariant Borel probability measure is conjugate to the universal odometer action [20, Theorem 1.5] (the case $d = 1$ was treated in [40]). One can already immediately see from the definition of the topology on spaces of actions on the Cantor set that the property of having a given action on a finite set as a factor is stable under perturbations whenever the acting group is finitely generated. This explains why in Theorem A we impose the property of weak mixing, which rules out nontrivial finite factors. On the other hand, Proposition 8.6 shows that, when X is the Cantor set, a generic action in $\mathrm{WA}(F_d, X)$ has the property that the homeomorphisms defined by the standard generators all factor onto the trivial action on the Cantor set and hence are very far from themselves being minimal. That $\mathrm{WA}(G * H, X)$ in Theorem A is nonempty can either be seen as a special case of a general phenomenon recorded in Proposition 8.4 or by means of the concrete examples constructed in Section 12.

In the case of F_d, if we further restrict our scope to actions that are minimal and spectrally aperiodic on generators then we have in fact been able to establish both the genericity of stable rank one and the meagreness of orbits, so that we are indeed capturing a relatively large class of actions, all rather different than the generic ones in $\mathrm{WA}(F_d, X)$, which as mentioned above are far from being minimal on generators. That this class is nonempty is illustrated by the examples in Section 12. We write $\mathrm{A}^*(F_d, X)$ for the set of all topologically free actions $F_d \curvearrowright X$ on the Cantor set that have an invariant Borel probability measure and are strictly ergodic (i.e., minimal and uniquely ergodic) and spectrally aperiodic (Definition 8.1) on each standard generator. In Theorem 10.4 we show that weak square divisibility is generic in $\mathrm{A}^*(F_d, X)$. Together with Theorem 4.7, this yields generic stable rank one:

Theorem D. *Let $d \geq 2$ and let X be the Cantor set. For a generic action in* $\mathrm{A}^*(F_d, X)$ *the reduced crossed product* $C(X) \rtimes_\lambda F_d$ *has stable rank one.*

The meagreness of orbits we record as follows and establish in Section 11.

Theorem E. *Let $d \geq 2$ and let X be the Cantor set. Then every orbit in* $\mathrm{A}^*(F_d, X)$ *under the conjugation action of the homeomorphism group of X is meagre.*

We begin in Section 2 by laying out some general definitions and basic dynamical background. Section 3 introduces the dynamical properties of square divisibility and weak square divisibility. These then appear as the operative hypotheses for Theorem 4.6 and 4.7 on stable rank one in Section 4. Section 5 establishes square divisibility for minimal actions of countably infinite discrete amenable groups on compact metrizable spaces under the assumptions of the URP and comparison (Theorem 5.7). With this at hand, we state and prove Theorem 6.1 on square divisibility in product actions in Section 6. Section 7 is devoted to the diagonal action machine that will be used in later sections for various purposes. After setting up the relevant spaces of actions in Section 8, we then turn to the proofs of Theorem 9.2 and 10.4 (generic weak square divisibility) in Sections 9 and 10 and of Theorem E (meagreness of orbits) in Section 11. To conclude we present some examples of squarely divisible actions of free groups in Section 12.

Acknowledgements. The authors were supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044/2-390685587, Mathematics Münster: Dynamics–Geometry–Structure, and by the SFB 1442 of the DFG. Portions of this work were carried out during visits of the third author to Queen's University Belfast in July 2025 as part of the INI program "C*-Algebras: Classification and Dynamical Constructions" and to Chongqing University in September 2025.

2. Notational conventions and preliminaries

Standing notation for groups and spaces. Throughout the paper G and H are countable discrete groups, to be explicitly subject to extra hypotheses depending on the circumstances. The identity element of a group will always be denoted e. By X and Y we always mean compact metrizable spaces, often to be explicitly specialized to the Cantor set.

We write $M(X)$ for the convex set of all Borel probability measures on X equipped with the weak* topology, under which it is compact. We use the notation $G \curvearrowright X$ to denote an action of G on X by homeomorphisms. Often we write the action via the simple concatenation $(s, x) \mapsto sx$ but when two or more actions are at play we will typically use symbols such as α to avoid confusion, so that the notation for the action becomes $(s, x) \mapsto \alpha_s x$. We similarly use sA or $\alpha_s A$ for the image of a set $A \subseteq X$ under s, and write LA for a set $L \subseteq G$ to mean $\bigcup_{s \in L} sA$. For sets $V \subseteq X$ and $E \subseteq G$ we write V^E for the intersection $\bigcap_{s \in E} s^{-1} V$.

When $G = \mathbb{Z}$ the action is generated by a single transformation $T : X \to X$ associated to the generator $1 \in \mathbb{Z}$, and so we will usually tacitly identify $\mathbb{Z}$-actions and single transformations (i.e., homeomorphisms of X) despite the abuse of notation this inevitably leads to. We will also use the notation $T \curvearrowright X$ for a transformation $T : X \to X$.

The action $G \curvearrowright X$ is *minimal* if there are no closed G-invariant subsets of X other than $\emptyset$ and X itself, or, equivalently, every G-orbit is dense. The *stabilizer* of a point $x \in X$ is the fixed point set $\{s \in G : sx = x\}$, which forms a subgroup of G. The action is *free* if the stabilizer of every point is trivial, and *topologically free* if the set of points with stabilizer equal to $\{e\}$ is dense (in which case it is actually a dense G_δ set). The action is *faithful* if for every $s \in G \setminus \{e\}$ there exists an $x \in X$ such that $sx \neq x$, i.e., the associated homomorphism of G into the homeomorphism group of X is injective.

The action $G \curvearrowright X$ is *transitive* if for all nonempty open sets $U, V \subseteq X$ there exists $s \in G$ such that $sU \cap V \neq \emptyset$, which is equivalent to the existence of a dense orbit (using that X is compact and metrizable). The action is *topologically mixing* if G is infinite and for all nonempty open sets $U, V \subseteq X$ there exists a finite subset $F \subseteq G$ such that $sU \cap V \neq \emptyset$ for all $s \in G \setminus F$, and *topologically weakly mixing* if for all nonempty open sets $U_1, U_2, V_1, V_2 \subseteq X$ there exists $s \in G$ such that $sU_1 \cap U_2 \neq \emptyset$ and $sV_1 \cap V_2 \neq \emptyset$, which is equivalent to the transitivity of the diagonal action $G \curvearrowright X \times X$ as given by $s(x, y) = (sx, sy)$. We also simply say *mixing* or *weakly mixing* if it is clear that the property we are referring to is not its measure-theoretic version.

For an action $G \curvearrowright X$, the set of all G-invariant measures in $M(X)$ is denoted $M_G(X)$, or $M_\alpha(X)$ if our action has a name α. This is a compact convex set whose extreme points are precisely the ergodic measures. The action is *uniquely ergodic* if $M_G(X)$ is a singleton, and *strictly ergodic* if it is minimal and uniquely ergodic.

In Sections 7, 9, and 10 we will have occasion to use some ergodic theory for p.m.p. (probability-measure-preserving) actions $G \curvearrowright (Z, \zeta)$. For this we will outsource most of the basic background and terminology to [45]. As in the topological setting, we identify a p.m.p. $\mathbb{Z}$-action on (Z, ζ) with the generating transformation T associated to $1 \in \mathbb{Z}$, which we also write as $T \curvearrowright (Z, \zeta)$. When we say that a p.m.p. action $G \curvearrowright (Z, \zeta)$ is *free* we mean that the set of points $x \in Z$ whose stabilizer $\{s \in G : sx = x\}$ is trivial has measure one. An action $G \curvearrowright X$ is *essentially free* if $M_G(X) \neq \emptyset$ and for every $\mu \in M_G(X)$ the p.m.p. action $G \curvearrowright (X, \mu)$ is free. It is easy to see that essentially free minimal actions are automatically topologically free. A p.m.p. action $G \curvearrowright (Z, \zeta)$ is *mixing* if G is infinite and for all measurable sets $A, B \subseteq Z$ and $\varepsilon > 0$ there exists a finite set $F \subseteq G$ such that $|\zeta(sA \cap B) - \zeta(A)\zeta(B)| < \varepsilon$ for all $s \in G \setminus F$, and it is *weakly mixing* if for all finite collections Ω of measurable subsets of Z and $\varepsilon > 0$ there exists an $s \in G$ such that $|\zeta(sA \cap B) - \zeta(A)\zeta(B)| < \varepsilon$ for all $A, B \in \Omega$. It is readily seen that if $G \curvearrowright X$ is an action and $\mu \in M_G(X)$ is a measure of full support such that $G \curvearrowright (X, \mu)$ is mixing (resp. weakly mixing) in the measure-theoretic sense then $G \curvearrowright X$ is topologically mixing (resp. topologically weakly mixing).

For a closed set $A \subseteq X$ and an open set $B \subseteq X$ we write $A \prec B$ (A is *(dynamically) subequivalent* to B) if there exist open sets $U_1, \dots, U_n \subseteq X$ and $s_1, \dots, s_n \in G$ such that $A \subseteq \bigcup_{i=1}^n U_i$ and the sets $s_1U_1, \dots, s_nU_n$ are pairwise disjoint subsets of B. For a set $F \subseteq G$ we write $A \prec_F B$ if $A \prec B$ and we can choose the group elements $s_1, \dots, s_n$ in the definition of subequivalence to belong to F. We also write $A \prec_\alpha B$ and $A \prec_{\alpha,F} B$ if the action α needs to be made explicit. If X is zero-dimensional and A and B are clopen subsets of X such that $A \prec B$ then one can take the sets $U_1, \dots, U_n$ in the definition of subequivalence to form a clopen partition of A [44, Proposition 3.5].

The action $G \curvearrowright X$ has *(dynamical) comparison* if for every closed set $A \subseteq X$ and open set $B \subseteq X$ satisfying $\mu(A) < \mu(B)$ for every $\mu \in M_G(X)$ one has $A \prec B$ [44, Definition 3.2]. One can also express this using pairs of open sets A and B by interpreting subequivalence in this case to mean $A_0 \prec B$ for every closed set $A_0 \subseteq A$.

A *tower* for the action $G \curvearrowright X$ is a pair (S, B) where S is a nonempty finite subset of G (the *shape*) and B is a nonempty subset of X (the *base*) such that the sets sB for $s \in S$ (the *levels*) are pairwise disjoint. The tower is open, clopen, etc., if the levels are open, clopen, etc. A *castle* is a finite collection $\{(S_k, V_k)\}_{k\in K}$ of towers such that the sets S_kV_k for $k \in K$ are pairwise disjoint. The *remainder* of the castle is the complement $X \setminus \bigsqcup_{k\in K} S_kV_k$. The castle is open, clopen, etc., if the towers are open, clopen, etc.

Let E be a finite subset of G and $\delta > 0$. For a set $F \subseteq G$ we write F^E for $\bigcap_{s\in E} s^{-1}F$. A finite set $F \subseteq G$ is said to be (E, δ)*-invariant* if $|F^{E\cup\{e\}}| \geq (1 - \delta)|F|$. The existence, for each finite set $E \subseteq G$ and $\delta > 0$, of an (E, δ)-invariant finite set $F \subseteq G$ is the Følner characterization of amenability for G. Another characterization of amenability for G is the nonemptiness of $M_G(X)$ for every action $G \curvearrowright X$ on a compact metrizable space. The class of amenable groups includes finite groups and Abelian groups and is closed under taking subgroups, quotients, extensions, and direct limits (the bootstrap class one thereby obtains comprises the *elementary amenable groups*). The canonical examples of nonamenable groups are the free groups $F_d = \langle a_1, \dots, a_d\rangle$ on $d \geq 2$ generators.

Associated to amenable groups and their actions are strong tiling properties, a couple of which we will exploit in Section 5 on the way to establishing Theorems A and D. One of these is the Ornstein–Weiss quasitiling theorem for Følner subsets of G (see Lemma 5.4). The other is the *uniform Rokhlin property (URP)* for an action $G \curvearrowright X$, which requires the existence, for every finite set $E \subseteq G$ and $\delta > 0$, of an open castle whose towers have (E, δ)-invariant shapes and whose remainder R satisfies $\sup_{\mu\in M_G(X)} \mu(R) < \delta$ [60, Definition 3.1]. This is similar to the stronger property of *almost finiteness*, which is more directly related to $\mathcal{Z}$-stability [44] and requires (i) that the remainder R instead be subequivalent to a set made up of a small (i.e., less than the given $\delta > 0$) proportion of levels in each tower, and (ii) that the tower levels additionally have small diameter with respect to a given compatible metric [44, Definition 8.2]. It is known that free actions of countably infinite discrete groups on compact metrizable spaces with finite covering dimension satisfy the URP, that free minimal actions of countably infinite Abelian groups on compact metrizable spaces satisfy the URP [59, Corollary E], that actions of elementary amenable groups and of finitely generated groups of subexponential growth on the Cantor set are almost finite [21, 47] (see also [58] for a more general result), and that a generic action of a fixed countably infinite amenable group on the Cantor set is almost finite [14, Theorem 4.2]. For free (or even essentially free [26]) actions of countably infinite amenable G, almost finiteness implies comparison [44] and is equivalent to it when the action has the small boundary property, which is automatic in the case that X is finite-dimensional [50].

For more on amenability and nonamenability, especially in connection with dynamical phenomena, see [45].

From an action $G \curvearrowright X$ one forms, in combination with the left regular representation $\lambda : G \to \mathcal{B}(\ell^2(G))$, the reduced crossed product C*-algebra $C(X) \rtimes_\lambda G$. This is a certain completion of the algebraic crossed product consisting of the sums $\sum_{s\in E} f_s u_s$ where E is a finite subset of G, the u_s are fixed unitaries indexed by G via a group homomorphism $s \mapsto u_s$, and the "coefficients" f_s are functions in $C(X)$, with the multiplication determined by the relation $u_s f u_s^{-1} = sf$ for all $s \in G$ and $f \in C(X)$ where $(sf)(x) = f(s^{-1}x)$ for $x \in X$. One can also form other completions, including a maximal one (the full crossed product), and if the group G is amenable, or more generally if the action is amenable, then the full and reduced crossed products will canonically coincide. A key technical feature of the reduced crossed product is the existence of a *faithful* conditional expectation $E : C(X) \rtimes_\lambda G \to C(X)$ satisfying $E(f_s u_s) = 0$ whenever $s \neq e$. Another important fact is that the reduced crossed product $C(X)\rtimes_\lambda G$ is simple as a C*-algebra whenever the action $G \curvearrowright X$ is minimal and topologically free [5].

A unital C*-algebra A has *stable rank one* if the set of elements in A that generate it as a left ideal is dense, or, equivalently, if the set of invertible elements in A is dense. Stable rank one implies stable finiteness, which in turn implies the existence of a quasitrace [37, 7]. A quasitrace on the reduced crossed product of an action $G \curvearrowright X$ restricts on $C(X)$ to integration with respect to a G-invariant Borel probability measure, so that $M_G(X) \neq \emptyset$ is a necessary condition for the crossed product to have stable rank one. For definitions and more on C*-algebras we refer the reader to [6, 11].

For fixed G and X we denote by $\mathrm{Act}(G,X)$ the space of all actions $G \curvearrowright X$. We endow $\mathrm{Act}(G,X)$ with the topology of pointwise compact-open convergence on individual group elements. If we fix a compatible metric d on X then a basis for this topology is given by the sets

$$U_{\alpha,E,\delta} = \{\beta \in \mathrm{Act}(G,X) : \sup_{x\in X} d(\beta_s x, \alpha_s x) < \delta \text{ for all } s \in E\}$$

where $\alpha \in \mathrm{Act}(G,X)$, E is a finite subset of G, and $\delta > 0$. Moreover we can endow $\mathrm{Act}(G,X)$ itself with a compatible metric by fixing an enumeration $s_1, s_2, \dots$ of G (assuming G is infinite and adjusting notation otherwise) and setting

$$\rho(\alpha,\beta) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{x\in X} d(\alpha_{s_k} x, \beta_{s_k} x).$$

The space $\mathrm{Act}(G,X)$ is complete under this metric and separable as a topological space, and so it is Polish.

In the case that X is the Cantor set, the topology on $\mathrm{Act}(G,X)$ also has as a basis the open sets

$$U_{\alpha,E,\mathscr{P}} = \{\beta \in \mathrm{Act}(G,X) : \beta_s A = \alpha_s A \text{ for all } A \in \mathscr{P} \text{ and } s \in E\}$$

where $\alpha \in \mathrm{Act}(G,X)$, E is a finite subset of G, and $\mathscr{P}$ is a clopen partition of X.

In Section 8 we will introduce certain subspaces of $\mathrm{Act}(G,X)$ in preparation for the genericity and meagreness results of Sections 9, 10, and 11. In the case of the genericity theorems we will need the following fact in the case that α is the product of β with some other action and h is the projection map onto the first coordinate. The idea is essentially the same as in the proof of [14, Theorem 4.2].

Proposition 2.1. *Suppose that X and Y are the Cantor set. Let $\alpha \in \mathrm{Act}(G,X)$ and $\beta \in \mathrm{Act}(G,Y)$ and suppose that there is a continuous surjection $h : Y \to X$ such that $\alpha \circ h = h \circ \beta$. Let $\{\mathscr{P}_i\}_{i\in I}$ be the net of all clopen partitions of X ordered by refinement. Then for every $i \in I$ there is a homeomorphism $g_i : Y \to X$ satisfying $g_i(h^{-1}(A)) = A$ for every $A \in \mathscr{P}_i$, and if $\{g_i\}_{i\in I}$ is any collection of such homeomorphisms then the actions β_i defined by $\beta_{i,s} = g_i \circ \beta_s \circ g_i^{-1}$ for $s \in G$ converge to α.*

Proof. Given an $i \in I$, for each $A \in \mathscr{P}_i$ both A and $h^{-1}(A)$ are nonempty clopen subsets (the latter thanks to surjectivity of h) and hence are homeomorphic to the Cantor set. Since $\mathscr{P}_i$ and $\{h^{-1}(A) : A \in \mathscr{P}_i\}$ are clopen partitions of X and Y, respectively, this permits us to construct a homeomorphism $g_i : Y \to X$ with the property that $g_i(h^{-1}(A)) = A$ for every $A \in \mathscr{P}_i$.

Suppose, for every $i \in I$, that g_i is a homeomorphism with this property. Let E be a finite subset of G and $\mathscr{P}$ a clopen partition of X. Given our description of the topology on $\mathrm{Act}(G,X)$

in terms of clopen partitions, in order to show that the actions β_i defined by $\beta_{i,s} = g_i \circ \beta_s \circ g_i^{-1}$ for $s \in G$ converge to α it suffices to demonstrate that there exists an $i_0 \in I$ such that $\beta_i \in U_{\alpha,E,\mathscr{P}}$ for all $i \geq i_0$.

Let $i_0 \in I$ be such that $\mathscr{P}_{i_0}$ is the join of the partitions $\{\alpha_{s^{-1}}A : A \in \mathscr{P}\}$ for $s \in E \cup \{e\}$. Let $i \geq i_0$ and $A \in \mathscr{P}$. Then for any $s \in E \cup \{e\}$ there exist $B_1, \dots, B_n \in \mathscr{P}_i$ such that $\alpha_s A = \bigsqcup_{j=1}^n B_j$ and hence

$$g_i(h^{-1}(\alpha_s A)) = g_i\bigg(\bigsqcup_{j=1}^n h^{-1}(B_j)\bigg) = \bigsqcup_{j=1}^n g_i(h^{-1}(B_j)) = \bigsqcup_{j=1}^n B_j = \alpha_s A.$$

We thereby obtain, for every $s \in E$,

$$\beta_{i,s}(A) = (g_i \circ \beta_s \circ g_i^{-1})(A) = g_i(\beta_s(h^{-1}(A))) = g_i(h^{-1}(\alpha_s A)) = \alpha_s A.$$

That is, $\beta_i \in U_{\alpha,E,\mathscr{P}}$, as required. □

Finally we observe the following stability of subequivalence under perturbations of the action. This will be needed in the proof of Lemma 3.13.

Proposition 2.2. *Let A and B be subsets of X with A closed and B open, and F a finite subset of G. Then the set of all $\alpha \in \mathrm{Act}(G,X)$ such that $A \prec_{\alpha,F} B$ is open.*

Proof. Let α be an action in $\mathrm{Act}(G,X)$ such that $A \prec_{\alpha,F} B$. Then there exist open sets $U_s \subseteq X$ for $s \in F$ such that $A \subseteq \bigcup_{s\in F} U_s$ and the sets $\alpha_s U_s$ for $s \in F$ are pairwise disjoint and contained in B. Fix a compatible metric d on X. For a set $D \subseteq X$ and $\varepsilon > 0$ write $D^\varepsilon = \{x \in D : d(x, D^c) > \varepsilon\}$. The open sets U_s^ε for $s \in F$ and $\varepsilon > 0$ cover A, and so by compactness there is a particular $\varepsilon > 0$ such that the sets U_s^ε for $s \in F$ cover A. Using compactness and the fact that the maps α_s for $s \in F$ are homeomorphisms, we can find a $\delta > 0$ such that $\alpha_s \overline{U_s^\varepsilon} \subseteq (\alpha_s U_s)^\delta$ for all $s \in F$. It is now easy to check that whenever $\beta \in U_{\alpha,F,\delta}$ one has $\beta_s \overline{U_s^\varepsilon} \subseteq \alpha_s U_s$ for every $s \in F$, so that $A \prec_{\beta,F} B$. □

3. Square divisibility

The notions of square divisibility (Definition 3.3) and weak square divisibility (Definition 3.10) that we introduce here are dynamical abstractions of part of the apparatus used to establish stable rank one in [53]. A key difference with [53] is that our definitions localize the subequivalences at play and thus do not make any blanket assumption of comparison. In the case of weak square divisibility this localization is set up so that we obtain a G_δ property in the space of actions (Proposition 3.14). This enables us to establish genericity results for stable rank one (Theorems 9.2 and 10.4) without having to directly confront the descriptive-set-theoretic nature of either comparison or stable rank one, which we do not know to be G_δ conditions. We do not even know if any of the actions in our genericity results satisfy comparison.

At the same time, our genericity results will rely on the fact that essentially free minimal actions of amenable groups on the Cantor set with comparison are (O_1, O_2, E)-squarely divisible in the sense of Definition 3.3 for all nonempty open sets O_1, O_2 and finite sets $e \in E \subseteq G$, as shown in Theorem 5.7 (such actions have the URP, independently of the comparison hypothesis [26]).

Definition 3.3, as well as Theorem 5.7 and the resulting stable rank one result (Theorem 4.6), deal with the general setting of compact metrizable X. This will yield, for example, a streamlined approach to establishing stable rank one for crossed products of free minimal $\mathbb{Z}^d$-actions, as originally shown by Li and Niu [53]. It is also conceivable that these results apply to some actions of nonamenable groups on higher-dimensional spaces, although the present paper will focus exclusively on the Cantor set once we cross the threshold into nonamenability in the later sections.

Definition 3.1. Let $G \curvearrowright X$ be an action. We say that the members of a collection $\{V_m\}_{m=1}^M$ of subsets of X are *pairwise equivalent* if there exist finitely many Borel subsets $\{C_k\}_{k\in K}$ of X satisfying $V_1 = \bigsqcup_{k\in K} C_k$ and, for every $k \in K$, elements $\{s_{k,m}\}_{m=1}^M$ of G with $s_{k,1} = e$ so that $V_m = \bigsqcup_{k\in K} s_{k,m}C_k$ and the sets $s_{k,m}C_k$ for $k \in K$ and $m = 1, \dots, M$ have pairwise disjoint closures in X.

Note that pairwise equivalence implies that the sets $\{V_m\}_{m=1}^M$ have pairwise disjoint closures.

Definition 3.2. Let $G \curvearrowright X$ be an action. For a set $E \subseteq G$, we say that the members of a collection $\mathscr{V}$ of subsets of X are *pairwise E-disjoint* if the sets $E\overline{V}$ for $V \in \mathscr{V}$ are pairwise disjoint.

Definition 3.3. Let $G \curvearrowright X$ be an action. Let O_1 and O_2 be open subsets of X and let E be a finite subset of G containing e. The action $G \curvearrowright X$ is (O_1, O_2, E)-*squarely divisible* if there exist an $n \in \mathbb{N}$, a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise E-disjoint open subsets of X, and, writing $V = \bigsqcup_{i,j=1}^n V_{i,j}$, an open set $U \subseteq X$ with $\partial V \subseteq U$ such that, defining $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = V^c$, and $B = \overline{V} \cap ((V \cap \overline{U}^c)^E)^c$, one has the following:

(i) $\overline{V_{i,1}} \prec O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ for every $i = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V_1 \cup B)^c$, and
(iii) $B \cup (\overline{U} \cap R) \prec O_2 \cap \overline{V \cup U}^c$.

Given a nonempty open set $O \subseteq X$, the action is O-*squarely divisible* if there exist three nonempty open sets $O_0 \subseteq O$ and $O_1, O_2 \subseteq X$ with pairwise disjoint closures such that

(iv) $\overline{O_1} \sqcup \overline{O_2} \prec O_0$, and
(v) the action is (O_1, O_2, E)-squarely divisible for all finite sets $E \subseteq G$ containing e.

Finally, the action is *squarely divisible* if it is O-squarely divisible for all nonempty open sets $O \subseteq X$.

Observe that B contains $\overline{U} \cap V$, and so $B \cup (\overline{U} \cap R)$ contains $\overline{U}$. Thus $B \cup (\overline{U} \cap R)$ is a thickening of the topological boundary between the tower and the remainder. This thickening involves both topological and group-theoretic aspects, namely the use of both U and E. Condition (i) says that the tower levels are small, condition (ii) says that the remainder is small, and condition (iii) says that the thickened boundary $B \cup (\overline{U} \cap R)$ is small. In each case this smallness is expressed with respect to certain parts of the tower structure in conjunction with an independently prescribed parts of the space X, namely O_1 and O_2. Figure 1 illustrates the definition in the case of a Cantor set, where one can take all of the sets at play to be clopen and the set U to be empty, as Proposition 3.9 will demonstrate.

The fact that we take the sets $V_{i,j}$ to be indexed by a square array is not essential for establishing Theorems 4.6 and 4.7, where a more general rectangular array would be sufficient, as long as it is not too thin in one direction. In our applications of Theorems 4.6 and 4.7, however, we can always get away with a square array, and so we have built it into the definition (and terminology) in order to take advantage of the notational economy it provides.

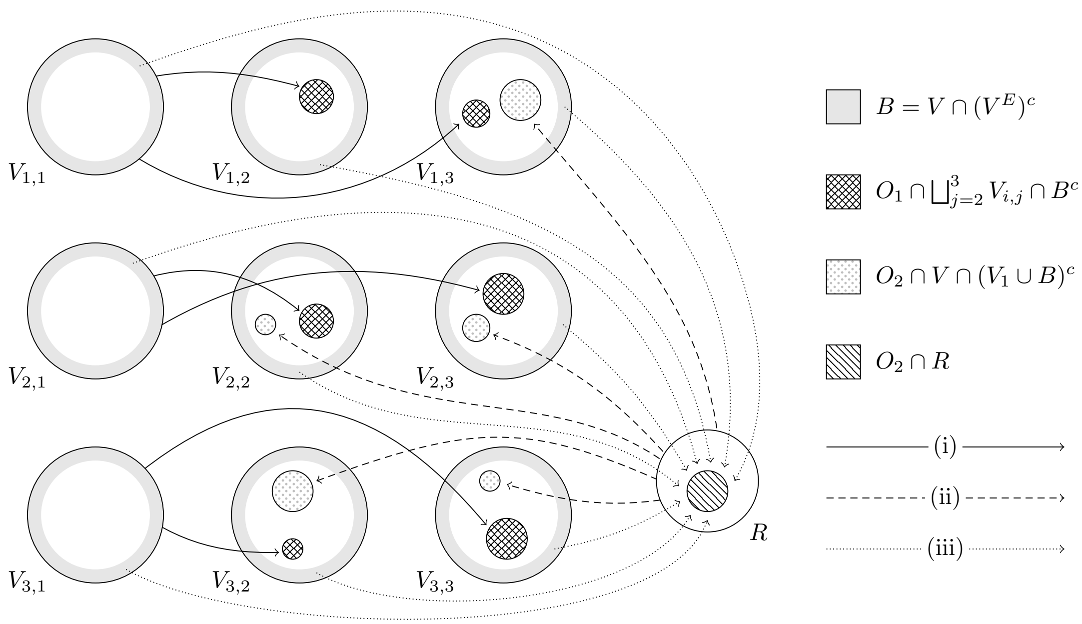

FIGURE 1. (O_1, O_2, E)-square divisibility ($n = 3$) in the Cantor set setting with $U = \emptyset$ (as in Proposition 3.9) and arrows indicating subequivalences.

The separation property $\overline{O_0} \cap (\overline{O_1} \sqcup \overline{O_2}) = \emptyset$ that is satisfied by the sets O_0, O_1, and O_2 in the definition of O-square divisibility is designed for the purpose of being able to construct a unitary, via Lemma 4.1, in the verification of stable rank one in Theorem 4.6. That the set $\overline{O_1} \sqcup \overline{O_2}$ be subequivalent to (and not merely contained in) the given open subset O provides us with a certain flexibility that is crucial for the proof of Theorem 6.1. This aspect of the definition is not needed, however, in the amenable setting of Theorem 5.7, where we show that minimal actions with the URP and comparison possess the stronger property of (O_1, O_2, E)-squarely divisibility for all nonempty open sets $O_1, O_2 \subseteq X$ and all finite sets $e \in E \subseteq G$, which also happens to be critical to the proof of Theorem 6.1. That this is indeed a stronger property we record in the following proposition.

Proposition 3.4. *Suppose that G is infinite. Let $G \curvearrowright X$ be a topologically free minimal action which is (O_1, O_2, E)-squarely divisible for all nonempty open sets $O_1, O_2, \subseteq X$ with $\overline{O_1} \cap \overline{O_2} = \emptyset$ and all finite sets $e \in E \subseteq G$. Then the action is squarely divisible.*

Proof. Let O be a nonempty open subset of X. Since G is infinite and the action is topologically free and minimal, X has no isolated points. Choose a point $x \in X$. By minimality, there exists $x_0 \in O$ distinct from x and $s \in G$ such that $sx = x_0$. Using the continuity of the action we can then find open neighbourhoods $O_x \subseteq X$ and $O_0 \subseteq O$ of x and x_0, respectively, such that $\overline{O_x} \cap \overline{O_0} = \emptyset$ and $sO_x \subseteq O_0$, the latter of which gives us $O_x \prec O_0$. Since X has no isolated points, there are open sets $O_1, O_2 \subseteq X$ with disjoint closures such that $\overline{O_1} \sqcup \overline{O_2} \subseteq O_x$. It now follows that $\overline{O_1} \sqcup \overline{O_2} \prec O_0$. Since by hypothesis the action is (O_1, O_2, E)-squarely divisible for all finite sets $E \subseteq G$ containing e, we deduce that the action is O-squarely divisible and hence squarely divisible. □

Remark 3.5. Knowing that a topologically free minimal action of an infinite group G is (O_1, O_2, E)-squarely divisible for all O_1, O_2, and E (as in Theorem 5.7) will imply stable rank one via Proposition 3.4 and Theorem 4.6, but one can reach this conclusion more directly by simply taking O_1 and O_2 to be subsets of O and dispensing with the construction of the unitary associated to the subequivalence $\overline{O_1} \sqcup \overline{O_2} \prec O_0 \subseteq O$ in the proof of Theorem 4.6.

In Definition 3.3 the set U and the subequivalences involving it are critical for allowing us to rotate the non-invertible element in the proof of Theorem 4.6 to an element b that is "spectrally null" around the boundary of V (this will allow us to build ersatz permutation unitaries using homotopies that do not spectrally interfere with b, so that these unitaries will act like genuine permutations on b and rotate it to something nilpotent). Not unrelated is the fact that when the space X is zero-dimensional the set U and its associated subequivalences also allow us to establish the simpler characterization of (O_1, O_2, E)-squarely divisibility in terms of clopen sets, recorded in Proposition 3.9 below. For this we need the following compactness principle for subequivalences.

Lemma 3.6. *Let $G \curvearrowright X$ be an action. Let $A \subseteq X$ be a closed set and $B \subseteq X$ an open set such that $A \prec B$. Then there exist an open set $V \supseteq A$ and an open set $W \subseteq B$ with $\overline{W} \subseteq B$ such that $\overline{V} \prec W$.*

Proof. By assumption there exist a collection $\{U_i\}_{i=1}^n$ of open subsets of X covering A and elements $s_1, \dots, s_n \in G$ such that the sets $s_i U_i$ are pairwise disjoint subsets of B. Using normality and compactness, we find for each $i = 1, \dots, n$ open subsets $V_i, W_i \subseteq X$ with

$$V_i \subseteq \overline{V_i} \subseteq W_i \subseteq \overline{W_i} \subseteq U_i$$

so that $A \subseteq \bigcup_{i=1}^n V_i$. Set $V = \bigcup_{i=1}^n V_i$ and $W = \bigsqcup_{i=1}^n s_i W_i$. Then $V \supseteq A$, $\overline{W} \subseteq B$, and $\overline{V} \prec W$, completing the proof. □

Lemma 3.7. *Let $G \curvearrowright X$ be an action on the Cantor set. Let O_1 and O_2 be open subsets of X and let E be a finite subset of G containing e, and suppose that the action is (O_1, O_2, E)-squarely divisible. Then there are closed sets $C_1 \subseteq O_1$ and $C_2 \subseteq O_2$ such that for all open sets O_1' and O_2' satisfying $C_1 \subseteq O_1' \subseteq O_1$ and $C_2 \subseteq O_2' \subseteq O_2$ the action is (O_1', O_2', E)-squarely divisible.*

Proof. Apply Lemma 3.6 to the definition of (O_1, O_2, E)-square divisibility to find open subsets $\{W_i\}_{i=1}^n$ of X such that $\overline{W_i} \subseteq O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ and $\overline{V_{i,1}} \prec W_i$ for each $i = 1, \dots, n$, and set $C_1 = \bigcup_{i=1}^n \overline{W_i}$. To define C_2 one similarly applies Lemma 3.6 only this time using the conditions (ii) and (iii) in the definition of (O_1, O_2, E)-square divisibility. □

Remark 3.8. It follows from the construction in the proof of Lemma 3.7 that if O_1 and O_2 are nonempty then so are C_1 and C_2.

Proposition 3.9. *Let $G \curvearrowright X$ be an action on the Cantor set. Let O_1 and O_2 be clopen subsets of X and let E be a finite subset of G containing e. The action $G \curvearrowright X$ is (O_1, O_2, E)-squarely divisible if and only if there exist an $n \in \mathbb{N}$, a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise E-disjoint clopen subsets of X, and, writing $V = \bigsqcup_{i,j=1}^n V_{i,j}$, $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = V^c$, and $B = V \cap (V^E)^c$, one has the following:*

(i) $V_{i,1} \prec O_1 \cap \bigsqcup_{j=2}^{n} V_{i,j} \cap B^c$ *for every* $i = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V_1 \cup B)^c$, *and*
(iii) $B \prec O_2 \cap R$.

Moreover, given a nonempty open set $O \subseteq X$*, the action is* O*-squarely divisible if and only if there exist nonempty disjoint clopen sets* $O_0, O_1, O_2 \subseteq X$ *with* $O_0 \subseteq O$ *such that*

(iv) $O_1 \sqcup O_2 \prec O_0$, *and*
(v) *the action is* (O_1, O_2, E)*-squarely divisible for all finite sets* $E \subseteq G$ *containing* e.

Finally, the action is squarely divisible if and only if it is O*-squarely divisible for all nonempty clopen sets* $O \subseteq X$.

Proof. Suppose that $G \curvearrowright X$ is (O_1, O_2, E)-squarely divisible. Then there exist $n \in \mathbb{N}$, a collection $\{\tilde{V}_{i,j}\}_{i,j}^n$ of pairwise equivalent and pairwise E-disjoint open subsets of X, and an open neighbourhood U of $\partial\tilde{V}$ satisfying the conditions in Definition 3.3 with $\tilde{V} = \bigsqcup_{i,j=1}^n \tilde{V}_{i,j}$, $\tilde{V}_1 = \bigsqcup_{i=1}^n \tilde{V}_{i,1}$, $\tilde{R} = \tilde{V}^c$, and $\tilde{B} = \overline{\tilde{V}} \cap ((\tilde{V} \cap \overline{U}^c)^E)^c$. Let $\{\tilde{C}_k\}_{k \in K}$ and $\{s_{k,i,j}\}_{i,j=1}^n$ for $k \in K$ be as in the definition of pairwise equivalence for the sets $\tilde{V}_{i,j}$. Since the collection $\{\tilde{V}_{i,j}\}_{i,j=1}^n$ is E-disjoint, we can use the continuity of the action, together with the compactness and normality of X, to find open sets $Y_{i,j} \supseteq \tilde{V}_{i,j}$ which are E-disjoint. In view of the separation properties these sets and group elements are required to satisfy, the zero-dimensionality of X implies the existence of clopen sets $\{C_k\}_{k \in K}$ such that $\tilde{C}_k \subseteq C_k$ for each $k \in K$ and the sets $s_{k,i,j}C_k$ are pairwise disjoint with $s_{k,i,j}C_k \subseteq (\tilde{V}_{i,j} \cup U) \cap Y_{i,j}$ for $k \in K$ and $i, j = 1, \dots, n$. Indeed, one first finds pairwise disjoint clopen sets $W_{k,i,j} \subseteq (\tilde{V}_{i,j} \cup U) \cap Y_{i,j}$ satisfying $s_{k,i,j}\overline{\tilde{C}_k} \subseteq W_{k,i,j}$ for $k \in K$ and $i, j = 1, \dots, n$ and then defines $C_k = \bigcap_{i,j=1}^n s_{k,i,j}^{-1} W_{k,i,j}$. For all $i, j = 1, \dots, n$ set $V_{i,j} = \bigsqcup_{k \in K} s_{k,i,j}C_k \supseteq \tilde{V}_{i,j}$. Then the sets $V_{i,j}$ are clopen, pairwise equivalent, and pairwise E-disjoint. Set $V = \bigsqcup_{i,j=1}^n V_{i,j} \supseteq \tilde{V}$, $V_1 = \bigsqcup_{i=1}^n V_{i,1} \supseteq \tilde{V}_1$, $R = V^c \subseteq \tilde{R}$, and $B = V \cap (V^E)^c$.

Since $\overline{\tilde{V}_{i,1}} \prec O_1 \cap \bigsqcup_{j=2}^n \tilde{V}_{i,j} \cap \tilde{B}^c$ for every $i = 1, \dots, n$ by condition (i) in the definition of square divisibility, Lemma 3.6 shows that we can take the sets C_k to be smaller if necessary so as to arrange for every $i = 1, \dots, n$ that

$$V_{i,1} \prec O_1 \cap \bigsqcup_{j=2}^{n} \tilde{V}_{i,j} \cap \tilde{B}^c \subseteq O_1 \cap \bigsqcup_{j=2}^{n} V_{i,j} \cap B^c.$$

Since we chose $s_{k,i,j}C_k$ to be contained in $\tilde{V}_{i,j} \cup U$ for all i, j, and k, we have $V_1 \subseteq \tilde{V}_1 \cup U$ and hence

$$\begin{aligned}\tilde{V} \cap (\tilde{V}_1 \cup \tilde{B})^c = \tilde{V}_1^c \cap (\tilde{V} \cap \overline{U}^c)^E &= (\tilde{V}_1 \cup U)^c \cap (\tilde{V} \cap \overline{U}^c)^E \\ &\subseteq V_1^c \cap V^E = V \cap (V_1 \cup B)^c.\end{aligned}$$

This implies

$$R \subseteq \tilde{R} \prec O_2 \cap \tilde{V} \cap (\tilde{V}_1 \cup \tilde{B})^c \subseteq O_2 \cap V \cap (V_1 \cup B)^c.$$

Finally, since we chose each $s_{k,i,j}C_k$ to be contained in $\tilde{V} \cup U$ we have $V \subseteq \tilde{V} \cup U$ and hence, applying this fact to get both of the inclusions below,

$$B \subseteq \tilde{B} \cup (\overline{U} \cap \tilde{R}) \prec O_2 \cap \overline{\tilde{V} \cup U}^c \subseteq O_2 \cap R.$$

This establishes the forward implication in the first part of the lemma statement. The reverse implication is immediate from the definition of (O_1, O_2, E)-square divisibility.

The reverse implication of the second part of the lemma statement is trivial, and so let us assume, in the other direction, that the action is O-squarely divisible. Then there are nonempty open sets $\tilde{O}_0 \subseteq O_0$ and $\tilde{O}_1, \tilde{O}_2 \subseteq X$ with pairwise disjoint closures such that $\overline{\tilde{O}_1} \sqcup \overline{\tilde{O}_2} \prec \tilde{O}_0$ and the action is $(\tilde{O}_1, \tilde{O}_2, E)$-squarely divisible for all finite subsets $E \subseteq G$ containing e. By Lemma 3.7 (see also Remark 3.8) and the zero-dimensionality of X, we can then find, for each finite set $E \subseteq G$ containing e, nonempty clopen sets $C_{1,E} \subseteq \tilde{O}_1$ and $C_{2,E} \subseteq \tilde{O}_2$ such that for all open sets O_1' and O_2' satisfying $C_{1,E} \subseteq O_1' \subseteq \tilde{O}_1$ and $C_{2,E} \subseteq O_2' \subseteq \tilde{O}_2$ the action is (O_1', O_2', E)-squarely divisible. Consider the (nonempty) clopen sets $O_1 := \bigcup_E C_{1,E}$ and $O_2 := \bigcup_E C_{2,E}$, where the (countable) unions range over all finite subsets $e \in E \subseteq G$. By construction, the action is (O_1, O_2, E)-squarely divisible for all finite subsets $e \in E \subseteq G$. Moreover, since $O_1 \subseteq \tilde{O}_1$ and $O_2 \subseteq \tilde{O}_2$ we have $O_1 \sqcup O_2 \prec \tilde{O}_0$. By Lemma 3.6 and the zero-dimensionality of X, there exists a nonempty clopen set $O_0 \subseteq \tilde{O}_0$ such that $O_1 \sqcup O_2 \prec O_0$. Finally we observe that O_0, O_1, O_2 are pairwise disjoint and that $O_0 \subseteq O$.

Since every nonempty open subset of X contains a clopen set, the reverse implication in the third part of the lemma statement is immediate, while the forward implication is trivial. □

We do not know if the set of squarely divisible actions in $\mathrm{Act}(G, X)$ is a G_δ. The problem is the universal quantification over E in condition (v) of Definition 3.3, which is nested within the universal quantification over O and existential quantification over O_0, O_1, and O_2. In order to remedy this deficiency so that we can establish the genericity results of Sections 9 and 10, we formulate the following weaker property in the Cantor set setting that will both be a G_δ condition in the space of actions and imply stable rank one. The definition appears somewhat ad hoc in the way that the set FEF appears, but this enables an order of quantification that permits us to construct the unitary in the first part of the proof of Theorem 4.7.

Definition 3.10. Let $G \curvearrowright X$ be an action on the Cantor set. Given a nonempty clopen set $O \subseteq X$ and a finite set $e \in E \subseteq G$, the action is *(O, E)-squarely divisible* if there exist pairwise disjoint nonempty clopen sets $O_0, O_1, O_2 \subseteq X$ with $O_0 \subseteq O$ and a finite symmetric set $e \in F \subseteq G$ with $O_1 \sqcup O_2 \prec_F O_0$ such that the action is (O_1, O_2, FEF)-squarely divisible. We say that the action is *weakly squarely divisible* if it is (O, E)-squarely divisible for every nonempty clopen set $O \subseteq X$ and finite set $e \in E \subseteq G$.

Remark 3.11. For an action $G \curvearrowright X$ on the Cantor set, we observe that O-square divisibility for a nonempty clopen set $O \subseteq X$ implies (O, E)-square divisibility for all finite sets $e \in E \subseteq G$ (see Proposition 3.9). In particular, square divisibility implies weak square divisibility. We do not know whether the converse holds.

To verify that weak square divisibility is a G_δ condition in the space of actions on the Cantor set, we first record a couple of lemmas.

Lemma 3.12. *Suppose X is the Cantor set. Let O_1, O_2 be nonempty clopen subsets of X and let $e \in E \subseteq G$ be a finite set. Then the set of all $\alpha \in \mathrm{Act}(G, X)$ that are (O_1, O_2, E)-squarely divisible is open.*

Proof. Let $\alpha \in \mathrm{Act}(G, X)$ be (O_1, O_2, E)-squarely divisible. By Proposition 3.9 we can find an $n \in \mathbb{N}$ and a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise E-disjoint clopen subsets of X such that the conditions (i)–(iii) in Proposition 3.9 are satisfied. Write $\Omega = \{1, \ldots, n\}^2$. The pairwise equivalence of the sets $\{V_{i,j}\}_{i,j=1}^n$ provides a finite clopen partition of $V_{1,1} = \bigsqcup_{k\in K} C_k$ and elements $s_{k,p} \in G$ for $k \in K$ and $p \in \Omega$ with $s_{k,(1,1)} = e$ for all $k \in K$ such that $V_p = \bigsqcup_{k\in K} \alpha_{s_{k,p}} C_k$. Let $\mathscr{P}$ be a finite clopen partition of X containing both $\{C_k\}_{k\in K}$ and $\{V_p\}_{p\in\Omega\setminus\{(1,1)\}}$, and let $F \subseteq G$ be a finite set containing $E \cup E^{-1} \cup \{s_{k,p}\}_{k\in K, p\in\Omega}$. For all $\beta \in U_{\alpha,F,\mathscr{P}}$ the sets $\{V_{i,j}\}_{i,j=1}^n$ are pairwise equivalent under β, with the equivalence being implementable by the same sets $\{C_k\}_{k\in K}$ given that $\alpha_{s_{k,p}} C_k = \beta_{s_{k,p}} C_k$ for all $k \in K$ and $p \in \Omega$. The sets $\{V_{i,j}\}_{i,j=1}^n$ are also E-disjoint under β since $\beta_s V_p = \alpha_s V_p$ for all $s \in E$ and $p \in \Omega$. Note also that, for the same reason, the boundary $B = V \cap (V^E)^c$, which in general depends on the action α, remains unchanged for β.

With the sets $V_{i,j}$ and B fixed in this way across all actions in $U_{\alpha,F,\mathscr{P}}$, we can now pass to a smaller open neighbourhood of α on which the subequivalences (i)–(iii) in Proposition 3.9 always hold (the argument is identical to the proof of Proposition 2.2, where we additionally had to control the group elements which implement the subequivalence). □

Lemma 3.13. *Suppose X is the Cantor set. Let O be a nonempty clopen subset of X and $e \in E$ a finite subset of G. Then the set $\mathscr{W}_{O,E}$ of all (O, E)-squarely divisible actions in $\mathrm{Act}(G, X)$ is open.*

Proof. Let $\alpha \in \mathscr{W}_{O,E}$. Then there exist nonempty disjoint clopen sets $O_0, O_1, O_2 \subseteq X$ with $O_0 \subseteq O$ and a finite symmetric set $e \in F \subseteq G$ with $O_1 \sqcup O_1 \prec_F O_0$ such that the action is (O_1, O_2, FEF)-squarely divisible. By Proposition 2.2, the subequivalence $O_1 \sqcup O_1 \prec_F O_0$ is stable under perturbation of α, and by Lemma 3.12 so is the property of (O_1, O_2, FEF)-square divisibility. It follows that α has a neighbourhood contained in $\mathscr{W}_{O,E}$, and so we conclude that $\mathscr{W}_{O,E}$ is open. □

In the proof below we use the well-known fact that the collection of clopen subsets of the Cantor set is countable. One can see this C*-algebraically by observing that the projections in $C(X)$ (which are precisely the indicator functions of clopen sets) are at norm distance 1 from each other. Since $C(X)$ is separable, this implies that there are only countably many projections.

Proposition 3.14. *Suppose X is the Cantor set. Then the set of all weakly squarely divisible actions in $\mathrm{Act}(G, X)$ is a G_δ.*

Proof. Write $\mathscr{O}$ for the countable collection of nonempty clopen subsets of X and $\mathscr{E}$ for the countable collection of finite subsets of G containing e. By Lemma 3.13, for every $O \in \mathscr{O}$ and $E \in \mathscr{E}$ the set $\mathscr{W}_{O,E}$ of all (O, E)-squarely divisible actions in $\mathrm{Act}(G, X)$ is open, and so the intersection $\bigcap_{O\in\mathscr{O}, E\in\mathscr{E}} \mathscr{W}_{O,E}$, which is equal to the set of all weakly squarely divisible actions in $\mathrm{Act}(G, X)$, is a G_δ. □

One obstruction to square divisibility and weak square divisibility is strong ergodicity. Recall that a p.m.p. action $G \curvearrowright (Z, \zeta)$ is *strongly ergodic* if for every sequence $(W_n)_{n\in\mathbb{N}}$ of Borel subsets of X with $\lim_{n\to\infty} \zeta(sW_n \Delta W_n) = 0$ for all $s \in G$ one has $\lim_{n\to\infty} \zeta(W_n)(1 - \zeta(W_n)) = 0$.

Proposition 3.15. *Let $G \curvearrowright X$ be a minimal squarely divisible action on an infinite compact metrizable space or a minimal weakly squarely divisible action on the Cantor set. Let $\mu \in M_G(X)$. Then the p.m.p. action $G \curvearrowright (X, \mu)$ is not strongly ergodic.*

Proof. The argument is the same in both cases, and so we assume that $G \curvearrowright X$ is a squarely divisible minimal action on an infinite compact metrizable space. This implies that μ is atomless and has full support. Thus for every $m \in \mathbb{N}$ there exists an open set $O_m \subseteq X$ with $0 < \mu(O_m) \leq 1/m$. Since G is countable, there exists an increasing sequence $(E_m)_{m=1}^{\infty}$ of finite subsets of G containing e such that $G = \bigcup_{m=1}^{\infty} E_m$. Let $m \in \mathbb{N}$. By assumption our action is O_m-squarely divisible, and so there exist nonempty open sets $O_{m,0} \subseteq O_m$ and $O_{m,1}, O_{m,2} \subseteq X$ with disjoint closures such that $\overline{O_{m,1}} \sqcup \overline{O_{m,2}} \prec O_{m,0}$ and the action is $(O_{m,1}, O_{m,2}, E_m)$-squarely divisible for each $m \in \mathbb{N}$. Denote by $\{V_{m,i,j}\}_{i,j=1}^{n_m}$ the sets obtained by this square divisibility and set $V_m = \bigsqcup_{i,j=1}^{n_m} V_{m,i,j}$. Since for every closed set $A \subseteq X$ and open set $B \subseteq X$ the relation $A \prec B$ clearly implies $\mu(A) \leq \mu(B)$, condition (iii) in Definition 3.3 implies that $\mu(V_m \setminus V_m^{E_m}) \leq 1/m$.

Set $\tilde{W}_m = \bigsqcup_{i=1}^{\lfloor n_m/2 \rfloor} \bigsqcup_{j=1}^{n_m} V_{m,i,j}$ and $W_m = \tilde{W}_m^{E_m}$. We claim that

$$\lim_{m\to\infty} \mu(sW_m \Delta W_m) = 0 \tag{3.1}$$

for all $s \in G$. Let $s \in G$ and $\varepsilon > 0$. Pick $M \in \mathbb{N}$ so that $1/M < \varepsilon/2$ and $s \in E_M$, and let $m \geq M$. We will show that $V_m^{E_m} \cap \tilde{W}_m \subseteq W_m$. Let $x \in V_m^{E_m} \cap \tilde{W}_m$ and $r \in E_m$. It suffices to verify that $rx \in \tilde{W}_m$. Since $x \in \tilde{W}_m$ and $V_{m,i,j}$ are E_m-disjoint, we have $rx \notin \bigsqcup_{i=\lfloor n_m/2 \rfloor+1}^{n_m} \bigsqcup_{j=1}^{n_m} V_{m,i,j}$. On the other hand we have $rx \in V_m$ by assumption and hence $rx \in \tilde{W}_m$, as we wanted to show. Observe also that $sW_m \cup W_m \subseteq \tilde{W}_m$. Using the invariance of μ, we obtain

$$\begin{aligned}
\mu(sW_m \Delta W_m) &\leq \mu(\tilde{W}_m \setminus W_m) + \mu(\tilde{W}_m \setminus sW_m) \\
&= 2\mu(\tilde{W}_m \setminus W_m) \\
&\leq 2\mu(\tilde{W}_m \setminus (V_m^{E_m} \cap \tilde{W}_m)) \\
&= 2\mu(\tilde{W}_m \setminus V_m^{E_m}) \\
&\leq 2\mu(V_m \setminus V_m^{E_m}) \leq \varepsilon,
\end{aligned}$$

which verifies (3.1).

We next check that

$$\lim_{m\to\infty} \mu(W_m) = \frac{1}{2}, \tag{3.2}$$

which will establish strong ergodicity. Let $\varepsilon > 0$. Since the sets $\{V_{m,i,j}\}_{i,j=1}^{n_m}$ are pairwise equivalent and μ is invariant, they all have equal μ-measure. Writing $R_m = V_m^c$, condition (ii) in Definition 3.3 implies that $\mu(R_m) \leq 1/m$. Together with condition (i) in Definition 3.3, we get

$$1 = n_m^2 \mu(V_{m,1,1}) + \mu(R_m) \leq \frac{n_m^2 + 1}{m}.$$

Let $M \in \mathbb{N}$ be such that $1/\sqrt{M-1} < \varepsilon/2$ and let $m \geq M$. Note that $1/n_m < \varepsilon/2$. Clearly $\mu(W_m) \leq 1/2$, and so we aim at showing that $\mu(W_m) \geq 1/2 - \varepsilon$. The pairwise equivalence of the sets $V_{m,i,j}$ implies that $2\mu(\tilde{W}_m) + n_m \mu(V_{m,1,1}) \geq \mu(V_m)$. Since $\mu(V_{m,1,1}) \leq 1/n_m^2$, we obtain

$$\mu(\tilde{W}_m) \geq \frac{1}{2}\mu(V_m) - \frac{1}{2} n_m \mu(V_{m,1,1}) \geq \frac{1}{2}\mu(V_m) - \frac{1}{2n_m} \geq \frac{1}{2}\mu(V_m) - \frac{\varepsilon}{4}.$$

Since $\mu(R_m) \leq 1/m < \varepsilon/2$ we have $\mu(V_m) \geq 1-\varepsilon/2$, and we conclude that $\mu(\tilde{W}_m) \geq 1/2-\varepsilon/2$. As shown in the first half of the proof, $\mu(\tilde{W}_m \setminus W_m) \leq \mu(V_m \setminus (V_m)^{E_m}) \leq 1/m < \varepsilon/2$. Thus

$$\mu(W_m) = \mu(\tilde{W}_m) - \mu(\tilde{W}_m \setminus W_m) \geq \frac{1}{2} - \varepsilon,$$

which verifies (3.2). □

Remark 3.16. Suppose that the action $G \curvearrowright (X,\mu)$ in Proposition 3.15 is free and ergodic, in which case the von Neumann algebra crossed product $L^\infty(X,\mu) \rtimes G$ is a II_1 factor. The fact that the action is not strongly ergodic implies that $L^\infty(X,\mu) \rtimes G$ has property Γ. Since for a given $m \in \mathbb{N}$ the sets $\{V_{m,i,j}\}_{i,j=1}^{n_m}$ in the proof are pairwise equivalent, one can use them to construct matrix embeddings $M_{n_m^2} \hookrightarrow L^\infty(X,\mu) \rtimes G$ that are almost unital in trace norm (up to the indicator function of the complement of their union). However, while the indicator functions of the set W_m are asymptotically central in $L^\infty(X,\mu) \rtimes G$ (thereby witnessing property Γ) the images of these matrix algebras, or of any of their noncommutative unital matrix subalgebras, cannot be made asymptotically central in general, as this would show that $L^\infty(X,\mu) \rtimes G$ has the McDuff property, which would imply that G is inner amenable [18, Proposition 4.1]. This would mean in particular that G cannot be one of the free groups F_d for $d \geq 2$, although these groups admit plenty of squarely divisible actions possessing an invariant Borel probability measure μ for which the corresponding p.m.p. action is free and ergodic, as shown in Propositions 12.6 and 12.13.

Example 3.17. By [82, 24], if G is countably infinite then there exists a free minimal action $G \curvearrowright X$ on the Cantor set which is universal for free p.m.p. actions of G, i.e., for every free measure-preserving action $G \curvearrowright (Z,\zeta)$ on a standard atomless probability space there exists a $\mu \in M_G(X)$ such that the p.m.p. actions $G \curvearrowright (X,\mu)$ and $G \curvearrowright (Z,\zeta)$ are measure conjugate.

Fix a countably infinite G and let (Y_0,ν_0) be a standard probability space which is nontrivial, i.e., which has no atom of full measure. It is well-known that the Bernoulli action $G \curvearrowright (Y_0^G,\nu_0^G)$ is free and mixing. If G is moreover assumed to be nonamenable then $G \curvearrowright (Y_0^G,\nu_0^G)$ is also strongly ergodic. By the previous paragraph, there exists a free minimal action $G \curvearrowright X$ on the Cantor set and $\mu \in M_G(X)$ such that the actions $G \curvearrowright (X,\mu)$ and $G \curvearrowright (Y_0^G,\nu_0^G)$ are measure conjugate. Then the p.m.p. action $G \curvearrowright (X,\mu)$ is free, mixing, and strongly ergodic. By Proposition 3.15, the action $G \curvearrowright X$ is neither squarely divisible nor weakly squarely divisible. Since μ has full support by minimality, we also see that $G \curvearrowright X$ is (topologically) mixing.

This shows in particular that for $G = F_d$ with $d \geq 2$ there exists a free minimal mixing action $F_d \curvearrowright X$ on the Cantor set that is neither squarely divisible nor weakly squarely divisible. This action belongs to the space $\mathrm{WA}(F_d,X)$ that is the subject of the generic weak square divisibility result in Section 9 and of Corollary B in the introduction (see Definition 8.2).

To round out this section we prove that square divisibility is an invariant of continuous orbit equivalence. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be two free actions and $\varphi : X \to Y$ a homeomorphism such that $\varphi(Gx) = H\varphi(x)$ for all $x \in X$. Then there are cocycles $\kappa : G \times X \to H$ and $\lambda : H \times Y \to G$ uniquely determined, as a consequence of freeness, by the equations $\varphi(gx) = \kappa(g,x)\varphi(x)$ and $\varphi^{-1}(hy) = \lambda(h,y)\varphi^{-1}(y)$. These cocycles are Borel but not in general continuous. When κ and λ are continuous we say that φ is a *continuous orbit equivalence* between the two actions. The actions are said to be *continuously orbit equivalent* if there exists a continuous orbit equivalence between them. Since continuity of the cocycles implies that they are locally constant, the theory of continuous orbit equivalence is primarily of interest for spaces that are zero-dimensional, or at least far from being connected.

Suppose that $\varphi : X \to Y$ is a continuous orbit equivalence between two free actions $G \overset{\alpha}{\curvearrowright} X$ and $H \overset{\beta}{\curvearrowright} Y$ with cocycle $\kappa : G \times X \to H$ as above. Since κ is continuous and H is discrete, for each $g \in G$ the image $\kappa(g, X)$ is finite in H. If $A \subseteq X$ is closed, $B \subseteq X$ is open, and F is a finite subset of G such that $A \prec_{\alpha,F} B$, then setting $L = \kappa(F, X)$ (which is finite by the continuity of κ) one has $\varphi(A) \prec_{\beta,L} \varphi(B)$, for if $\{U_g\}_{g\in F}$ is an open cover of A such that the sets $\alpha_g U_g$ for $g \in F$ are pairwise disjoint subsets of B then setting $D_{g,h} = \{x \in X : \kappa(g, x) = h\}$ for $g \in G$ and $h \in H$ the collection $\{\varphi(D_{g,h} \cap U_g)\}_{g\in F,\, h\in\kappa(g,X)}$ is an open cover of $\varphi(A)$ such that the sets $\beta_h\varphi(D_{g,h} \cap U_g)$ for $g \in F$ and $h \in \kappa(g, X)$ are pairwise disjoint subsets of $\varphi(B)$. Using these observations one can show the following.

Proposition 3.18. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be two free actions which are continuously orbit equivalent. Suppose that $G \curvearrowright X$ is squarely divisible. Then $H \curvearrowright Y$ is squarely divisible.*

Proof. Let $\varphi : X \to Y$ be a continuous orbit equivalence between the two actions, with corresponding cocycle maps $\kappa : G \times X \to H$ and $\lambda : H \times Y \to G$. Let $O \subseteq Y$ be a nonempty open set. We want to show that $H \curvearrowright Y$ is O-squarely divisible. Since $G \curvearrowright X$ is $\varphi^{-1}(O)$-squarely divisible, one can find nonempty open sets $O_0, O_1, O_2 \subseteq X$ with disjoint closures such that $O_0 \subseteq \varphi^{-1}(O)$, $\overline{O_1} \sqcup \overline{O_2} \prec O_0$, and $G \curvearrowright X$ is (O_1, O_2, E)-squarely divisible for all finite sets $e \in E \subseteq G$. Considering the sets $\varphi(O_0), \varphi(O_1), \varphi(O_2) \subseteq Y$, it is clear, based on the discussion before the proposition statement, that $H \curvearrowright Y$ will be squarely divisible if we manage to show that it is $(\varphi(O_1), \varphi(O_2), F)$-squarely divisible for all finite sets $e \in F \subseteq H$.

Let $e \in F \subseteq H$ be a finite set. Consider the finite set $E := \lambda(F, Y) \subseteq G$. Then $e \in E$, and by assumption $G \curvearrowright X$ is (O_1, O_2, E)-squarely divisible. Let $\{\tilde{V}_{i,j}\}_{i,j=1}^n$ be open sets in X implementing this square divisibility, and denote by R and B the remainder and boundary, respectively (as in Definition 3.3). Set $V_{i,j} = \varphi(\tilde{V}_{i,j})$ for $i, j = 1, \dots, n$ and write $\tilde{R}$ and $\tilde{B}$ for the corresponding remainder and boundary. We claim that these sets witness $(\varphi(O_1), \varphi(O_2), F)$-square divisibility for $H \curvearrowright Y$. It is a straightforward computation to verify that the E-disjointness of the sets $\{\tilde{V}_{i,j}\}_{i,j=1}^n$ implies that the sets $\{V_{i,j}\}_{i,j=1}^n$ are F-disjoint. It is more subtle to check that the sets $\{V_{i,j}\}_{i,j=1}^n$ are pairwise equivalent. To do that, let $\{\tilde{C}_k\}_{k\in K}$ be Borel subsets of X and $\{s_{k,i,j}\}_{k\in K, i,j=1,\dots,n}$ elements of G such that $\tilde{V}_{1,1} = \bigsqcup_{k\in K} \tilde{C}_k$ and $\tilde{V}_{i,j} = \bigsqcup_{k\in K} s_{k,i,j}\tilde{C}_k$ for all $i, j = 1, \dots, n$. Fix $i, j \in \{1, \dots, n\}$, with $(i, j) \neq (1, 1)$. As before define $D_{g,h} = \{x \in X : \kappa(g, x) = h\}$ for $g \in G$ and $h \in H$. One checks that

$$V_{1,1} = \bigsqcup_{k\in K} \bigsqcup_{h\in\kappa(s_{k,i,j},X)} \varphi(\tilde{C}_k \cap D_{s_{k,i,j},h}),$$

and

$$V_{i,j} = \bigsqcup_{k\in K} \bigsqcup_{h\in\kappa(s_{k,i,j},X)} h\varphi(\tilde{C}_k \cap D_{s_{k,i,j},h}).$$

Thus the partition $\mathcal{C}_{i,j} := \{\varphi(\tilde{C}_k \cap D_{s_{k,i,j},h}) : k \in K,\, h \in \kappa(s_{k,i,j}, X)\}$ implements an equivalence between $V_{1,1}$ and $V_{i,j}$. To obtain a pairwise equivalence between the sets $\{V_{i,j}\}_{i,j=1}^n$ we need however to find a partition of $V_{1,1}$ that works simultaneously for all $i, j = 1, \dots, n$. Consider $\mathcal{C} = \bigvee_{i,j=1,\dots,n} \mathcal{C}_{i,j}$, i.e., the common refinement. This is a partition of $V_{1,1}$ into Borel sets which is easily seen to implement the desired pairwise equivalence. The discussion preceding the proposition statement now completes the proof modulo the fact that $B \subseteq \varphi(\tilde{B})$, which follows from the inclusion $\varphi(A^E) \subseteq \varphi(A)^F$ for every $A \subseteq X$. □

We do not know whether weak square divisibility for free actions on the Cantor set is preserved under continuous orbit equivalence.

4. Stable rank one

Our goal here is to show that, for topologically free minimal actions, stable rank one is a consequence of square divisibility (Theorem 4.6) and also, in the Cantor set setting, of weak square divisibility (Theorem 4.7). This we do by recasting some of the main ideas of [53], which in turn have precedents in [71, 73, 77, 1, 55]. The common strategy originates in the work of Rørdam in [71]. Rørdam's first insight was that if a simple unital C*-algebra is finite then in order to approximate a non-invertible element a by invertible elements one may assume, by performing a unitary rotation and perturbation, that $ad = da = 0$ for some nonzero positive element d. Under favourable circumstances (such as $\mathcal{Z}$-stability [73] or, as we will show, square divisibility) the element a may then be rotated by unitaries to produce a nilpotent element, which can then be perturbed to something invertible by adding a small scalar multiple of the identity. One can then rotate such a perturbation back to an (invertible) element that approximates the original non-invertible element, yielding stable rank one. In the dynamical framework one furthermore needs the element d above to be of a special form, and for this we will appeal to a result in [53].

The argument in the Cantor setting is much simpler as we can build unitaries and partial isometries directly from clopen sets and group elements, without having to negotiate boundaries. The reader seeking to get the quickest grasp of the basic mechanisms at play is thus advised to start with the proof of Theorem 4.7.

Given an action $G \curvearrowright X$, the embedding of $C(X)$ into the C*-algebra $B(X)$ of bounded Borel functions on X extends canonically to an embedding of $C(X) \rtimes_\lambda G$ into $B(X) \rtimes_\lambda G$. When convenient we will accordingly view $C(X) \rtimes_\lambda G$ as sitting inside the larger C*-algebra $B(X) \rtimes_\lambda G$. To be specific, we will find it useful to be able to express certain multiplicative relationships using indicator functions of Borel sets.

The following is a version of Lemma 3.2 in [53] that is tailored to the setting of dynamical subequivalence and relies on the same functional calculus construction.

Lemma 4.1. *Let $G \curvearrowright X$ be an action. Let $A \subseteq X$ be a closed set and $B \subseteq X$ an open set such that $A \cap B = \emptyset$ and $A \prec B$. Let $A_0 \subseteq X$ be an open set with $A \subseteq A_0$. Then there are a closed set $B^- \subseteq B$, an open set $A^+ \subseteq X$ satisfying $A \subseteq A^+ \subseteq A_0$ and $A^+ \cap B^- = \emptyset$, and a unitary $u \in C(X) \rtimes_\lambda G$ such that the following hold:*

- *(i)* $u1_{A^+ \cup D} = 1_{B^- \sqcup D} u 1_{A^+ \cup D}$ *for every Borel set $D \subseteq X$ with $D \cap B^- = \emptyset$, and in particular* $u1_{A^+} = 1_{B^-} u 1_{A^+}$,
- *(ii) the same as (i) with u^* in place of u,*
- *(iii)* $1_{A_0 \cup B^-}(u-1) = u - 1 = (u-1)1_{A_0 \cup B^-}$, *and in particular* $u1_C = 1_{A_0 \cup B^- \cup C} u 1_C$ *and* $1_C u = 1_C u 1_{A_0 \cup B^- \cup C}$ *for every Borel set $C \subseteq X$,*
- *(iv)* $1_C u = u 1_C = 1_C$ *where* $C = (A_0 \cup B^-)^c$.

Proof. Since $A \prec B$ we can find open sets $U_1, \dots, U_n \subseteq X$ and elements $s_1, \dots, s_n \in G$ such that $A \subseteq \bigcup_{i=1}^n U_i \subseteq A_0$ and the sets $s_i U_i$ for $i = 1, \dots, n$ are pairwise disjoint and all contained in B. We may assume, by using the compactness of A to shrink the sets U_i slightly, that the closures of the sets s_iU_i for $i = 1, \dots, n$ are pairwise disjoint and contained in B. Define B^- to be the union of these closures.

Again using the compactness of A as above, we can find open sets $W_i \subseteq \overline{W_i} \subseteq U_i$ for $i = 1, \dots, n$ such that $A \subseteq \bigcup_{i=1}^n W_i$. Proceeding as in the usual construction of a partition of unity for the open cover $\{U_1, \dots, U_n, (X \setminus \bigcup_{i=1}^n \overline{W_i})\}$ of X, we can find $h_1, \dots, h_n \in C(X, [0,1])$ such that $h_i = 0$ off of U_i and the function $h := \sum_{i=1}^n h_i$ takes values in $[0,1]$ and is equal to 1 on $\bigcup_{i=1}^n \overline{W_i}$.

A separation argument applied to the closed disjoint sets A and B^- allows us to find an open set $A^+ \subseteq X$ such that $A \subseteq A^+ \subseteq \overline{A^+} \subseteq \bigcup_{i=1}^n W_i$ and $\overline{A^+} \cap B^- = \emptyset$. Taking $f \in C(X, [0,1])$ to be a function with $f|_{B^-} = 0$ and $f|_{A^+} = 1$ and multiplying each h_i with f, we may assume that $h|_{B^-} = 0$ and $h|_{A^+} = 1$. Note that $s_ih_i = 0$ off of $s_iU_i \subseteq B^-$ and therefore $s_ih_i \perp s_jh_j$ for $i \neq j$, and $s_ih_i \perp h$ for $i = 1, \dots, n$.

Set $v = \sum_{i=1}^n h_i^{1/2} u_{s_i}^*$. Then

$$vv^* = \sum_{i,j} h_i^{1/2} u_{s_i}^* u_{s_j} h_j^{1/2} = \sum_{i,j} u_{s_i}^* (s_ih_i^{1/2})(s_jh_j^{1/2}) u_{s_j} = \sum_i u_{s_i}^* (s_ih_i) u_{s_i} = h$$

while

$$v^*v = \sum_{i,j} u_{s_i} h_i^{1/2} h_j^{1/2} u_{s_j}^* = \sum_{i,j} (s_ih_i^{1/2}) u_{s_i} u_{s_j}^* (s_jh_j^{1/2}),$$

from which we see that $vv^* \leq 1$ and $v^*v \perp vv^*$. We are now in the situation of [53, Lemma 3.2], which shows, writing w for the self-adjoint element $vv^* + v^*v$ and g for the continuous function on $[0, \infty)$ defined by $g(t) = \sin(\pi t/2)/\sqrt{t}$ for $t > 0$ and $g(0) = 0$, that the element

$$u := \cos\left(\frac{\pi}{2} w\right) + g(vv^*)v - g(v^*v)v^*$$

is unitary.

We now verify (i). Let $D \subseteq X$ be a Borel set satisfying $D \cap B^- = \emptyset$. By construction each $s_ih_i^{1/2}$ vanishes off of B^-, and so we have

$$v1_{A^+ \cup D} = \sum_i h_i^{1/2} u_{s_i}^* 1_{A^+ \cup D} = \sum_i u_{s_i}^* (s_ih_i^{1/2}) 1_{A^+ \cup D} = 0.$$

The vanishing of each $s_ih_i^{1/2}$ off of B^- also yields

$$1_{B^-} v^* = \sum_i 1_{B^-} u_{s_i} h_i^{1/2} = \sum_i 1_{B^-} (s_ih_i^{1/2}) u_{s_i} = v^*$$

and hence $1_{B^- \sqcup D} v^*v = v^*v$, which shows, using that $g(0) = 0$, that

$$1_{B^- \sqcup D}\, g(v^*v) v^* 1_{A^+ \cup D} = g(v^*v) v^* 1_{A^+ \cup D}.$$

Finally, since $v^*v1_{A^+ \cup D} = 0$, $vv^* \perp v^*v$, and $h|_{A^+} = 1$ we have

$$\cos\left(\frac{\pi}{2} w\right) 1_{A^+ \cup D} = \cos\left(\frac{\pi}{2} vv^*\right) 1_{A^+ \cup D} = \cos\left(\frac{\pi}{2} h\right) 1_{A^+ \cup D} = \cos\left(\frac{\pi}{2} h\right) 1_D,$$

while the fact that 1_D commutes with $\cos(\pi h/2)$ implies

$$1_{B^- \sqcup D} \cos\left(\frac{\pi}{2} w\right) 1_{A^+ \cup D} = \cos\left(\frac{\pi}{2} w\right) 1_{A^+ \cup D}.$$

From these computations we obtain (i).

To see that (i) holds also for

$$u^* = \cos\left(\frac{\pi}{2}w\right) + v^*g(vv^*) + vg(v^*v),$$

we observe that $1_{B^-}v^* = v^*$ and hence $1_{B^-\sqcup D}v^* = v^*$ so that

$$1_{B^-\sqcup D}v^*g(vv^*)1_{A^+\cup D} = v^*g(vv^*)1_{A^+\cup D},$$

while from the equalities $v^*v1_{A^+\cup D} = 0$ and $g(0) = 0$ we obtain

$$1_{B^-\sqcup D}vg(v^*v)1_{A^+\cup D} = 0 = vg(v^*v)1_{A^+\cup D}.$$

This yields $1_{B^-\sqcup D}u^*1_{A^+\cup D} = u^*1_{A^+\cup D}$, as desired.

To establish (iii) one first checks that $1_{A_0\cup B^-}v = v$ (using that each h_i vanishes off of A_0) and $1_{A_0\cup B^-}v^* = v^*$ (using that each s_ih_i is supported in B^-). It follows that $1_{A_0\cup B^-}$ acts like a unit on w. Since $\cos(0) = 1$ we have

$$1_{A_0\cup B^-}\left(\cos\left(\frac{\pi}{2}w\right) - 1\right) = \cos\left(\frac{\pi}{2}w\right) - 1$$

and so altogether we obtain $1_{A_0\cup B^-}(u-1) = u - 1 = (u-1)1_{A_0\cup B^-}$. In particular, if $C \subseteq X$ is any Borel subset then $1_{A_0\cup B^-\cup C}$ acts like a unit on $u-1$. Therefore $u = (u-1)1_{A_0\cup B^-\cup C} + 1$, and so $1_Cu = 1_Cu1_{A_0\cup B^-\cup C}$. Similarly, $u1_C = 1_{A_0\cup B^-\cup C}u1_C$. This verifies condition (iii).

It remains to show that $1_Cu = u1_C = 1_C$ where $C = (A_0\cup B^-)^c$. Again following the supports of the functions involved, it is immediate that $vv^* \perp 1_C$ and $v^*v \perp 1_C$, from which we obtain $v \perp 1_C$ using the C*-identity. This in turn implies $1_Cu = u1_C = \cos(0)1_C = 1_C$. □

We make the following two observations in the framework of Lemma 4.1.

Remark 4.2. Suppose that X is zero-dimensional and that $A, B \subseteq X$ are disjoint clopen sets with $A \prec B$. Then the picture in Lemma 4.1 becomes much simpler and we can avoid the use of the functional calculus to construct the unitary u. By [44, Proposition 3.5] there exist a clopen partition $A = \bigsqcup_{i=1}^n A_i$ and elements $s_1, \dots, s_n \in G$ such that $B^- := \bigsqcup_{i=1}^n s_iA_i \subseteq B$. The element

$$u := 1_{X\setminus(A\sqcup B^-)} + \sum_{i=1}^n 1_{s_iA_i}u_{s_i} + \sum_{i=1}^n 1_{A_i}u_{s_i}^* \in C(X) \rtimes_\lambda G$$

is a self-adjoint unitary such that $u1_A = 1_{B^-}u$ and $1_Cu = u1_C = 1_C$ where $C = (A \sqcup B^-)^c$.

Remark 4.3. Suppose that $A_1, \dots, A_n \subseteq X$ are closed sets and $A_{1,0}, \dots, A_{n,0}, B_1, \dots, B_n \subseteq X$ are open sets such that $A_i \prec B_i$ and $A_i \subseteq A_{i,0}$ for all i and $(A_{i,0} \cup B_i) \cap (A_{j,0} \cup B_j) = \emptyset$ for all $i \neq j$. For every i let u_i be a unitary as given by Lemma 4.1 with respect to the subequivalence $A_i \prec B_i$, with the set A_0 in the statement of the lemma taken to be $A_{i,0}$. Then $u_iu_j = u_ju_i$, for all i, j. Indeed, setting $Y_i = A_{i,0} \cup B_i^-$ we have $Y_i \cap Y_j = \emptyset$ for $i \neq j$, and by condition (iv) in Lemma 4.1 we have $u_i = 1_{Y_i}u_i1_{Y_i} + 1_{Y_i^c}$. One can now easily check that $u_iu_j = 1_{Y_i}u_i1_{Y_i} + 1_{Y_j}u_j1_{Y_j} + 1_{Y_i^c}1_{Y_j^c} = u_ju_i$.

The following lemmas record a couple of simple facts that will be of use in the homotopy constructions in the proof of Theorem 4.6.

Lemma 4.4. *Let $C \subseteq V \subseteq X$ be open sets such that C is relatively clopen in V. Let $h \in C(X)$ be a function which vanishes off of V. Then $h1_C \in B(X)$ actually belongs to $C(X)$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to some point $x \in X$. If $x \in C$, then since C is open in X one has $x_n \in C$ for all sufficiently large n, so that

$$\lim_{n\to\infty} h(x_n)1_C(x_n) = \lim_{n\to\infty} h(x_n) = h(x) = h(x)1_C(x).$$

If $x \notin C$ but $x \in \partial C$, then from the fact that V is open and the complement of C is relatively open in V we infer that $x \notin V$, so that $\lim_{n\to\infty} h(x_n) = h(x) = 0$ and hence $\lim_{n\to\infty} h(x_n)1_C(x_n) = 0 = h(x)1_C(x)$. Finally, if $x \in X \setminus \overline{C}$ then $x_n \in X \setminus \overline{C}$ for all sufficiently large n, in which case $\lim_{n\to\infty} h(x_n)1_C(x_n) = 0 = h(x)1_C(x)$. ☐

Lemma 4.5. *Let A be a unital C*-algebra such that $C(X)$ is a C*-subalgebra of A sharing the same unit. Let $\{e_{i,j}\}_{i,j=1}^n$ be matrix units in A and let $\{g_{i,j}\}_{i,j=1}^n$ be functions in $C(X)$ such that $g_{i,j}e_{k,\ell} = e_{k,\ell}g_{i,j}$ for all $i, j, k, \ell = 1, \dots, n$. Suppose that for every $x \in X$ the element $\sum_{i,j=1}^n g_{i,j}(x)e_{i,j}$ is a unitary in $D := C^*(\{e_{i,j}\}_{i,j=1}^n) \cong M_n$. Then the element $a := \sum_{i,j=1}^n g_{i,j}e_{i,j}$ is a unitary in $C^*(\{g_{i,j}e_{i,j}\}_{i,j=1}^n)$.*

Proof. Thanks to the commutation relations in the hypotheses, one has a $*$-homomorphism

$$\iota \times \iota : C^*(1, \{g_{i,j}\}_{i,j=1}^n) \otimes D \to A$$

defined on elementary tensors by $(\iota\times\iota)(a\otimes b) = ab$. Recall that there is a natural $*$-isomorphism

$$C(X) \otimes D \cong C(X, D)$$

which maps $f \otimes e_{i,j}$ to the function $x \mapsto f(x)e_{i,j}$ for all $f \in C(X)$ and $i, j = 1, \dots, n$. In particular, the element $\sum_{i,j=1}^n g_{i,j} \otimes e_{i,j}$ is mapped to the function $x \overset{h}{\mapsto} \sum_{i,j=1}^n g_{i,j}(x)e_{i,j}$. By our hypotheses, both h^*h and hh^* are equal to the constant function $\sum_{i=1}^n e_{i,i}$. It follows that the element $b := \sum_{i,j=1}^n g_{i,j} \otimes e_{i,j}$ satisfies $b^*b = bb^* = 1 \otimes (\sum_{i=1}^n e_{i,i})$, and applying the $*$-homomorphism $\iota \times \iota$ to this equation yields $a^*a = aa^* = \sum_{i=1}^n e_{i,i}$, as desired. ☐

Theorem 4.6. *Let $G \curvearrowright X$ be a topologically free minimal action with $M_G(X) \neq \emptyset$, and suppose that it is squarely divisible. Then the reduced crossed product $C(X) \rtimes_\lambda G$ has stable rank one.*

Proof. By topological freeness and minimality, $C(X)\rtimes_\lambda G$ is simple [5]. It is also stably finite, and in particular finite, since any invariant Borel probability measure gives rise to a faithful tracial state via composition with the canonical conditional expectation onto $C(X)$, with faithfulness being a consequence of simplicity.

Let a be a non-invertible element of $C(X) \rtimes_\lambda G$. Following the strategy of Rørdam [71] as described in the beginning of this section, we will first apply perturbation and unitary rotation to a so as to produce a two-sided zero divisor of a suitable type and then perform a series of further perturbations and unitary rotations in order to produce a nilpotent element. Since a nilpotent element can be approximated arbitrarily well by invertible elements through the addition of nonzero scalar multiples of the unit, this will establish stable rank one.

In the general setting of simple unital finite C*-algebras, Rørdam shows that a non-invertible element b can be perturbed and unitarily rotated to an element b' satisfying $b'd = db' = 0$ for some nonzero positive element d, which is the strengthened form of zero division that is enables one, under favourable circumstances, to unitarily rotate to a nilpotent element [71, Proposition 3.2]. In our case we need a refinement of this principle for reduced crossed products that holds under the conditions of finiteness and simplicity that we observed in the first paragraph and allows us to assume, through perturbation and unitary rotation, that $a1_O = 1_Oa = 0$ for some nonempty open set $O \subseteq X$ [53, Proposition 6.2]. We note that Proposition 6.2 of [53] assumes freeness but the proof still works if one relaxes this to topological freeness, which is what we need here.

With the set O now at hand, by the definition of square divisibility we can find an open set $O_0 \subseteq O$ and nonempty open sets $O_1, O_2 \subseteq X$ with $\overline{O_1} \cap \overline{O_2} = \emptyset$, $O_0 \cap (\overline{O_1} \sqcup \overline{O_2}) = \emptyset$, and $\overline{O_1} \sqcup \overline{O_2} \prec O_0$ such that the action is (O_1, O_2, E)-squarely divisible for all finite sets $e \in E \subseteq G$.

Set $Q = \overline{O_1} \sqcup \overline{O_2}$ for brevity. Since $Q \cap O_0 = \emptyset$ we may apply Lemma 4.1 to find a unitary $r \in C(X) \rtimes_\lambda G$, a closed set $O_0^- \subseteq O_0$, and an open set $Q^+ \supseteq Q$ such that $r1_{Q^+} = 1_{O_0^-} r 1_{Q^+}$ and $1_{Q^+} r = 1_{Q^+} r 1_{O_0^-}$.

Take a $t \in C(X, [0,1])$ such that $t(x) = 0$ for all $x \in Q$ and $t(x) = 1$ for all $x \in X \setminus Q^+$. Then $rar(1-t) = rar1_{Q^+}(1-t) = ra1_{O_0^-} r 1_{Q^+}(1-t) = 0$ and, similarly, $(1-t)rar = 0$. We may therefore replace a with rar and assume that $a(1-t) = (1-t)a = 0$.

Now if a' is any element in the algebraic crossed product $C(X) \rtimes_{\mathrm{alg}} G$ then the element $a'' := ta't \in C(X) \rtimes_{\mathrm{alg}} G$ satisfies $a''1_Q = 1_Q a'' = 0$ and

$$\|a'' - a\| = \|t(a' - a)t\| \leq \|a' - a\|.$$

Since we may choose such an a' so that $\|a' - a\|$ is as small as we wish, by replacing a once again, this time with a'', we may assume that it satisfies $a1_Q = 1_Q a = 0$ and has the form $a = \sum_{s \in E} f_s u_s$ for some finite symmetric set $e \in E \subseteq G$. We now aim to show that a can be multiplied on the left and right by suitable unitaries in $C(X) \rtimes_\lambda G$ so as to obtain a nilpotent element.

By our choice of O_1 and O_2 as given by the definition of square divisibility, there exist an $n \in \mathbb{N}$, a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise E-disjoint open subsets of X, and, writing $V = \bigsqcup_{i,j=1}^n V_{i,j}$, an open set $U \subseteq X$ with $\partial V \subseteq U$ such that, defining $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = V^c$, and $B = \overline{V} \cap ((V \cap \overline{U}^c)^E)^c$, we have

(i) $\overline{V_{i,1}} \prec O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ for every $i = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V_1 \cup B)^c$, and
(iii) $B \cup (\overline{U} \cap R) \prec O_2 \cap \overline{V \cup U}^c$.

Note that in each of the subequivalences in (i), (ii), and (iii) the set on the left hand side is closed and disjoint from the open set on the right hand side.

Using a separation argument and the fact that the sets $\overline{V_{i,j}}$ are pairwise disjoint, we can find mutually disjoint open neighbourhoods $\mathcal{V}_i \supseteq \overline{V_{i,1}}$ for $i = 1, \dots, n$ such that for each i we have $\mathcal{V}_i \subseteq V_{i,1} \cup U$ and $\mathcal{V}_i \cap (\overline{V} \setminus \overline{V_{i,1}}) = \emptyset$, that is, $\mathcal{V}_i \subseteq (\overline{V}^c \cap U) \sqcup \overline{V_{i,1}}$. Applying Lemma 4.1 with respect to these neighbourhoods for the subequivalences in (i), for each $i = 1, \dots, n$ we obtain a unitary $u_i \in C(X) \rtimes_\lambda G$ and a closed set $Y_{i,1} \subseteq O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ such that $u_i 1_{V_{i,1} \cup D} = 1_{Y_{i,1} \sqcup D} u_i 1_{V_{i,1} \cup D}$ whenever $D \subseteq X$ is a Borel set satisfying $D \cap Y_{i,1} = \emptyset$. Moreover, $1_{(\overline{V}^c \cap U) \sqcup \overline{V_{i,1}} \sqcup Y_{i,1}}$ acts like a unit on $u_i - 1$, and $1_C u_i = u_i 1_C = 1_C$ for Borel sets $C \subseteq (\overline{V} \cup U^c) \cap \overline{V_{i,1}}^c \cap Y_{i,1}^c$. By the choice of the open neighbourhoods $\mathcal{V}_i \supseteq \overline{V_{i,1}}$ and Remark 4.3, we have $u_i u_j = u_j u_i$, for all $1 \leq i, j \leq n$. Writing $Y_1 = \bigsqcup_{i=1}^n Y_{i,1}$ and $u = u_1 u_2 \cdots u_n$, we obtain:

(u.1) $u1_{V_1 \cup D} = 1_{Y_1 \cup D} u 1_{V_1 \cup D}$ whenever $D \subseteq X$ is a Borel subset such that $D \cap Y_1 = \emptyset$,
(u.2) $1_{(\overline{V}^c \cap U) \sqcup \overline{V_1} \sqcup Y_1}$ acts like a unit on $u - 1$, and
(u.3) $u1_C = 1_C = 1_C u$ where $C = (\overline{V} \cup U^c) \cap \overline{V_1}^c \cap Y_1^c$.

In the case of the subequivalence (ii), we apply Lemma 4.1 with respect to some open neighbourhood R_0 of R that is contained in both $R \cup U$ and the complement of the closed set Y_1, which is disjoint from R. We thereby obtain a closed subset $Y_2 \subseteq O_2 \cap V \cap (V_1 \cup B)^c$, an open set $R^+ \subseteq X$ satisfying $R \subseteq R^+ \subseteq R_0$, and a unitary element $v \in C(X) \rtimes_\lambda G$ such that

(v.1) $v1_{R^+ \cup D} = 1_{Y_2 \sqcup D} v 1_{R^+ \cup D}$, whenever $D \subseteq X$ is a Borel set such that $D \cap Y_2 = \emptyset$,
(v.2) $1_{R_0 \cup Y_2}$ acts like a unit on $(v-1)$, and
(v.3) $1_C v = 1_C = v 1_C$ where $C = (R_0 \cup Y_2)^c$.

In the case of the subequivalence (iii), we apply Lemma 4.1 with respect to some open neighbourhood B_0 of $B \cup (\overline{U} \cap R)$ that is disjoint from $(Y_1 \cup Y_2)$. We thereby obtain an open set B^+ with $B \cup (\overline{U} \cap R) \subseteq B^+ \subseteq B_0$, a closed set $Y_3 \subseteq O_2 \cap \overline{V \cup U}^c$, and a unitary w such that

(w.1) $w1_{B^+ \cup D} = 1_{Y_3 \sqcup D} w 1_{B^+ \cup D}$ and $1_{B^+ \cup D} w = 1_{B^+ \cup D} w 1_{Y_3 \sqcup D}$ (since the first equation also holds for w^*) whenever $D \subseteq X$ is a Borel set such that $D \cap Y_3 = \emptyset$,
(w.2) $1_{B_0 \cup Y_3}$ acts like a unit on $(w-1)$, and
(w.3) $1_C w = 1_C = w 1_C$ where $C = (B_0 \cup Y_3)^c$.

Define $b = u^* wauv$. By partitioning the unit in $B(X) \rtimes_\lambda G$ as $1_R + \sum_{i,j=1}^n 1_{V_{i,j}}$ and multiplying b on either side by the projections in this sum we can represent b as an $(n^2+1) \times (n^2+1)$ "matrix". We will verify that, with respect to this matrix decomposition, b takes the following form (assuming $n = 3$ for the purpose of illustration), where the first row and column correspond to the set R and the remaining rows and columns correspond to the sets $V_{i,j}$ as ordered lexicographically with respect to the pairs i, j:

$$\left[\begin{array}{c|ccc|ccc|ccc} 0 & 0 & * & * & 0 & * & * & 0 & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{array}\right].$$

A matrix over $\mathbb{C}$ of this form can be multiplied on the left and right by permutation matrices with 1 in the upper left entry so as to produce a strictly upper triangular (and hence nilpotent) matrix, and indeed the proof will ultimately be completed by finding proxies z_1 and z_2 for these permutation matrices in the unitary group of $C(X) \rtimes_\lambda G$ so that $z_1 b z_2$ has a similar strictly upper triangular matrix representation and hence is nilpotent. Note that if all of the sets in the definition of square divisibility and in the subequivalences and equivalences at play therein were clopen, then we could directly interpret these permutation matrices as elements of $C(X) \rtimes_\lambda G$ (using the equivalences between the sets $V_{i,j}$, via Remark 4.2, to define off-diagonal matrix units in the lower right $n^2 \times n^2$ block, as we will do in the proof of Theorem 4.7) and thereby avoid the homotopy constructions that will occupy a large part of the reminder of the proof.

These homotopies, which will be used to define the unitaries z_1 and z_2 that rotate b to a nilpotent element, will need to be "invisible" to b, and so what we need in fact is a slightly more refined representation of b that takes into account the left zero division of b across the boundary of V, namely

$$b = 1_R b 1_V + \sum_{\substack{1\le i,j,j'\le n \\ j,j'\neq 1}} 1_{V_{i,j}\setminus R^+} b 1_{V_{i,j'}\setminus R^+}, \tag{4.1}$$

the point being that the slightly smaller projection $1_{V_{i,j}\setminus R^+}$ appears here instead of $1_{V_{i,j}}$. To verify (4.1), we begin by noting, using that $a1_{Y_2} = 0$, that

$$auv1_{R^+} \overset{\text{(v.1)}}{=} au1_{Y_2}v1_{R^+} \overset{\text{(u.3)}}{=} a1_{Y_2}v1_{R^+} = 0,$$

which shows that $b1_{R^+} = 0$. This in particular yields the zero leftmost column in the matrix representation of b, for which we just need $b1_R = 0$. Using the fact that $V_1 \cap Y_2 = Y_2 \cap Y_1 = \emptyset$ and that $a1_{Y_1\sqcup Y_2} = 0$, we also have

$$auv1_{V_1} \overset{\text{(v.1)}}{=} au1_{V_1\sqcup Y_2}v1_{V_1} \overset{\text{(u.1)}}{=} a1_{Y_1\sqcup Y_2}u1_{V_1\sqcup Y_2}v1_{V_1} = 0,$$

which accounts for the other n columns of zeros. Moreover, since $Y_1 \subseteq (B_0 \cup Y_3)^c$ and $1_{Y_1}a = 0$ we see that

$$1_{V_1}u^*wa \overset{\text{(u.1)}}{=} 1_{V_1}u^*1_{Y_1}wa \overset{\text{(w.3)}}{=} 1_{V_1}u^*1_{Y_1}a = 0$$

and hence $1_{V_1}b = 0$, which accounts for the n rows of zeros in the lower right $n^2 \times n^2$ submatrix.

Now let $1 \le i, i' \le n$ with $i \neq i'$ and let $2 \le j, j' \le n$. Writing $p = (i, j)$ and $q = (i', j')$ with V_p and V_q interpreted accordingly as $V_{i,j}$ and $V_{i',j'}$, let us check that $1_{V_p}b1_{V_q} = 0$, which will in particular account for the remaining zeros in the matrix representation of b. We do this by verifying the equalities (4.2), (4.3), (4.4), and (4.5) and then applying these in a chain-like way to obtain (4.6).

First observe, since $1_{R_0\cup Y_2\cup V_q}$ acts like a unit on $(v - 1)$ by (v.2), that

$$v1_{V_q} = 1_{R_0\cup Y_2\cup V_q}v1_{V_q}. \tag{4.2}$$

Next, setting $A = R_0 \cup Y_2 \cup V_q \cup V_{i',1}$ for brevity, we note that the fact that $1_{(\overline{V}^c\cap U)\sqcup\overline{V_{i',1}}\sqcup Y_{i',1}}$ acts like a unit on $(u_{i'} - 1)$ means that so does $1_{A\cup Y_{i',1}}$ due to the inclusion $(\overline{V}^c\cap U)\sqcup\overline{V_{i',1}}\sqcup Y_{i',1} \subseteq A \cup Y_{i',1}$, and thus

$$u_{i'}1_{R_0\cup Y_2\cup V_q} = 1_{A\cup Y_{i',1}}u_{i'}1_{R_0\cup Y_2\cup V_q}.$$

Given that the sets $Y_{k,1}$ are pairwise disjoint and $Y_{k,1} \cap A = \emptyset$ for all $k \neq i'$ (since $Y_{k,1} \subseteq (V_{k,2} \cup \cdots \cup V_{k,n}) \cap B^c$), we therefore obtain

$$\begin{aligned} u1_{R_0\cup Y_2\cup V_q} &= u_1 \cdots u_{i'-1}u_{i'+1} \cdots u_{n-1}u_n 1_{A\cup Y_{i',1}}u_{i'}1_{R_0\cup Y_2\cup V_q} \\ &= u_1 \cdots u_{i'-1}u_{i'+1} \cdots u_{n-1}1_{A\cup Y_{i',1}\cup Y_{n,1}}u_n 1_{A\cup Y_{i',1}}u_{i'}1_{R_0\cup Y_2\cup V_q}. \end{aligned}$$

Proceeding recursively in this way, we conclude that

$$u1_{R_0\cup Y_2\cup V_q} = 1_{A\cup Y_1}u1_{R_0\cup Y_2\cup V_q}. \tag{4.3}$$

Next we reexpress $a1_{A\cup Y_1}$. Note first that since the sets $\{V_{i,j}\}_{i,j=1}^n$ are E-disjoint, we clearly have $s(V_q \cup V_{i',1}) \subseteq R \cup V_q \cup V_{i',1}$ for all $s \in E$. Moreover, $(R \cup B)^c = (V \cap \overline{U}^c)^E \subseteq s(V \cap U^c)$ for all $s \in E$, using that $E = E^{-1}$. In other words, $s(R \cup U) \subseteq R \cup B$ for all $s \in E$. Putting these facts together, and recalling that $R_0 \subseteq R \cup U$, we have

$$s(R_0 \cup V_q \cup V_{i',1}) \subseteq R \cup B \cup V_q \cup V_{i',1}$$

for all $s \in E$. Since $a1_{Y_i} = 1_{Y_i}a = 0$ for $i = 1, 2, 3$, it follows that

$$a1_{A\cup Y_1} = a1_{R_0\cup V_q\cup V_{i',1}} = \sum_{s\in E} f_s u_s 1_{R_0\cup V_q\cup V_{i',1}} \tag{4.4}$$
$$= \sum_{s\in E} 1_{s(R_0\cup V_q\cup V_{i',1})} f_s u_s$$
$$= 1_{R\cup B\cup V_q\cup V_{i',1}} a 1_{R_0\cup V_q\cup V_{i',1}}$$
$$= 1_{(R\cup B\cup V_q\cup V_{i',1})\setminus Y_3} a 1_{A\cup Y_1}.$$

Finally, we reexpress $u^*1_{(R\setminus U)\sqcup V_q\sqcup V_{i',1}}$. Using (w.1) along with the fact that $(U\cap R)\cup B \subseteq B^+$, and $Y_3 \subseteq R \setminus U$, we compute that

$$w1_{(R\cup B\cup V_q\cup V_{i',1})\setminus Y_3} = w1_{((R\setminus U)\cup(U\cap R)\cup B\cup V_q\cup V_{i',1})\setminus Y_3}$$
$$= 1_{(R\setminus U)\sqcup V_q\sqcup V_{i',1}} w1_{((R\setminus U)\cup(U\cap R)\cup B\cup V_q\cup V_{i',1})\setminus Y_3}.$$

Note that for every $k \neq i'$, we have $u_k^*1_{V_q\sqcup V_{i',1}} = 1_{V_q\sqcup V_{i',1}}$. Also, $1_{(\overline{V}^c\cap U)\cup\overline{V_{i',1}}\cup Y_{i',1}}$ acts as a unit on $(u_{i'} - 1)$, and so does $1_{(R\cap U)\cup\bigsqcup_{j=1}^n \overline{V_{i',j}}}$ since $(\overline{V}^c \cap U) \cup \overline{V_{i',1}} \cup Y_{i',1} \subseteq (R\cap U) \cup \bigsqcup_{j=1}^n \overline{V_{i',j}}$. It follows that

$$u^*1_{V_q\cup V_{i',1}} = u_{i'}^*1_{V_q\cup V_{i',1}} = 1_{(R\cap U)\cup\bigsqcup_{j=1}^n \overline{V_{i',j}}} u_{i'}^* 1_{V_q\cup V_{i',1}}.$$

On the other hand, since $R\setminus U \subseteq (\overline{V}\cup U^c)\cap\overline{V_1}^c\cap Y_1^c$ condition (u.3) implies that $u^*1_{R\setminus U} = 1_{R\setminus U}$. In conjunction with the display above, this gives us

$$u^*1_{(R\setminus U)\sqcup V_q\sqcup V_{i',1}} = 1_{R\cup\bigsqcup_{j=1}^n \overline{V_{i',j}}} u^*1_{(R\setminus U)\sqcup V_q\sqcup V_{i',1}}. \tag{4.5}$$

We can now apply (4.2), (4.3), (4.4), and (4.5) in sequence to obtain

$$b1_{V_q} = u^*wauv1_{V_q} \tag{4.6}$$
$$= 1_{R\cup\bigsqcup_{j=1}^n \overline{V_{i',j}}} u^*1_{(R\setminus U)\sqcup V_q\sqcup V_{i',1}} w1_{(R\cup B\cup V_q\cup V_{i',1})\setminus Y_3} a1_{A\cup Y_1} u1_{R_0\cup Y_2\cup V_q} v1_{V_q}.$$

As $V_p\cap(R\cup\bigsqcup_{j=1}^n \overline{V_{i',j}}) = \emptyset$, we conclude that $1_{V_p}b1_{V_q} = 0$. We have thus verified that the matrix representation of b has the desired form, i.e., that (4.1) holds with $1_{V_{i,j}}$ in place of $1_{V_{i,j}\setminus R^+}$.

To obtain (4.1) itself, it remains now to check, given $1 \le i, j \le n$ and writing $p = (i, j)$, that $1_{R^+\cap V_p}b = b1_{R^+\cap V_p} = 0$. The second equality follows from $b1_{R^+} = 0$, which we observed at the outset. To verify that $1_{R^+\cap V_p}b = 0$, we will compute that $1_{R^+\cap V_p}u^*wa = 0$. We saw earlier that $1_{V_1}b = 0$, and so we may assume $j \neq 1$. Since $R^+ \cap V_p \subseteq (R \cup U) \cap V = U \cap V \subseteq B$ and hence $R^+ \cap V_p \subseteq \overline{V} \cap \overline{V_1}^c \cap Y_1^c$, from (u.3) we obtain $u1_{R^+\cap V_p} = 1_{R^+\cap V_p}$ and hence $1_{R^+\cap V_p}u^* = 1_{R^+\cap V_p}$. By (w.1) we have $1_Bw = 1_Bw1_{Y_3}$ and thus, since $R^+ \cap V_p \subseteq B$ as just observed,

$$1_{R^+\cap V_p}w = 1_{R^+\cap V_p}w1_{Y_3}.$$

It follows that

$$1_{R^+\cap V_p}u^*wa = 1_{R^+\cap V_p}wa = 1_{R^+\cap V_p}w1_{Y_3}a = 0,$$

as desired.

To set up the homotopies involved in the construction of z_1 and z_2, we first define some matrix units $e_{p,q}$ in $B(X) \rtimes_\lambda G$ and bump functions f and h in $C(X) \rtimes_\lambda G$. Set $\Omega = \{1, \dots, n\}^2$. By the equivalence of the sets $V_{i,j}$, which as above we also write as V_p where $p = (i, j) \in \Omega$, we can find a finite partition $\{C_k\}_{k\in K}$ of $V_{1,1}$ which is relatively clopen in $V_{1,1}$ and tuples $s_k = (s_{k,p})_{p\in\Omega}$ of elements of G for $k \in K$ such that $V_p = \bigsqcup_{k\in K} s_{k,p}C_k$ for all $p \in \Omega$, with $s_{k,(1,1)}$ equal to the identity of G for all $k \in K$, and the closures in X of the sets $s_{k,p}C_k$ for $p \in \Omega$ and $k \in K$ are pairwise disjoint. We can then find open neighbourhoods $W_{k,p} \supseteq s_{k,p}\overline{C_k}$ with pairwise disjoint closures. For each $k \in K$ define the open set $\tilde{C}_k = \bigcap_{p\in\Omega} s_{k,p}^{-1}W_{k,p} \supseteq \overline{C_k}$. Then the sets $s_{k,p}\tilde{C}_k$ for $k \in K$ and $p \in \Omega$ have pairwise disjoint closures in X. Define $\tilde{V}_p = \bigsqcup_{k\in K} s_{k,p}\tilde{C}_k$. Note that this gives a partition of $\tilde{V}_p$ which is clopen in the relative topology on $\tilde{V}_p$. Moreover, the sets $\tilde{V}_p$ for $p \in \Omega$ have pairwise disjoint closures in X. In particular, by Lemma 4.4 it follows that

(•) if $q \in C(X)$ is any function that vanishes off of $\tilde{V}_{1,1}$ then $q1_{\tilde{C}_k} \in C(X)$ for all $k \in K$.

For $p \in \Omega$ define $d_p = \sum_{k\in K} u_{s_{k,p}} 1_{\tilde{C}_k}$. Then $d_p^*d_p = 1_{\tilde{V}_{1,1}}$ and $d_pd_p^* = 1_{\tilde{V}_p}$, so that d_p is a partial isometry in $B(X) \rtimes_\lambda G$. Observe also that $d_p^*d_q = 0$ whenever $p \neq q$.

For $p, q \in \Omega$ define $e_{p,q} = d_pd_q^* \in B(X) \rtimes_\lambda G$. Note that $e_{p,q}^* = e_{q,p}$ and $\sum_{p\in\Omega} e_{p,p} = \sum_{p\in\Omega} 1_{\tilde{V}_p} = 1_{\tilde{V}}$ where $\tilde{V} = \bigsqcup_{p\in\Omega} \tilde{V}_p$. Moreover, using Kronecker delta notation,

$$e_{p,q}e_{r,s} = d_pd_q^*d_rd_s^* = \delta_{q,r}d_pd_r^*d_rd_s^* = \delta_{q,r}d_p1_{\tilde{V}_{1,1}}d_s^* = \delta_{q,r}d_pd_s^* = \delta_{q,r}e_{p,s}.$$

We conclude that the elements $e_{p,q}$ for $p, q \in \Omega$ are partial isometries in $B(X) \rtimes_\lambda G$ forming a set of matrix units.

Next we use the partial isometries d_p to build a bump function h which will be equal to 1 on V and 0 off of $\tilde{V}$ and a second bump function f which will be equal to 1 on $V \setminus R^+$ and 0 off of V. To define z_1 and z_2, we will perform homotopies over the second of these two sets using f and then a patching operation over the first one using h to achieve global unitarity.

To construct h, choose an $h_1 \in C(X, [0, 1])$ satisfying $h_1|_{V_{1,1}} = 1$ and $h_1|_{X\setminus\tilde{V}_{1,1}} = 0$, which is possible since $\overline{V_{1,1}} \subseteq \tilde{V}_{1,1}$. We claim that $\{d_ph_1d_p^*\}_{p\in\Omega}$ is a collection of pairwise orthogonal functions in $C(X)$. Indeed

$$d_ph_1d_p^* = \sum_{k,j\in K} u_{s_{k,p}}1_{\tilde{C}_k}h_11_{\tilde{C}_j}u_{s_{j,p}}^* = \sum_{k\in K} u_{s_{k,p}}1_{\tilde{C}_k}h_1u_{s_{k,p}}^* = \sum_{k\in K} 1_{s_{k,p}\tilde{C}_k}(s_{k,p}h_1),$$

and the last sum, which is clearly supported on $\tilde{V}_p$, belongs to $C(X)$ seeing that $1_{\tilde{C}_k}h_1 \in C(X)$ by (•). Set $h = \sum_{p\in\Omega} d_ph_1d_p^* \in C(X)$, which is a positive contraction supported on $\tilde{V}$. For all $p, q \in \Omega$ we have

$$\begin{aligned} he_{p,q} = \sum_{r\in\Omega} d_rh_1d_r^*d_pd_q^* &= d_ph_1d_p^*d_pd_q^* \\ &= d_ph_1d_q^* \\ &= d_pd_q^*d_qh_1d_q^* \\ &= d_pd_q^* \cdot \sum_{r\in\Omega} d_rh_1d_r^* = e_{p,q}h, \end{aligned}$$

and also

$$he_{p,q} = d_ph_1d_q^* = \sum_{k,j\in K} u_{s_{k,p}}1_{\tilde{C}_k}h_11_{\tilde{C}_j}u_{s_{j,q}}^* = \sum_{k\in K} u_{s_{k,p}}1_{\tilde{C}_k}h_1u_{s_{k,q}}^*,$$

which by (•) shows that $he_{p,q}$ belongs to $C(X) \rtimes_\lambda G$. Observe moreover that, for every $x \in V$, taking $p \in \Omega$ and $k \in K$ such that $x \in s_{k,p}C_k$ and using the fact that $s_{k,p}^{-1}(x) \in C_k \subseteq V_{1,1}$ and $h_1|_{V_{1,1}} = 1$, we have

$$h(x) = (d_p h_1 d_p^*)(x) = (s_{k,p}h_1)(x) = 1,$$

that is, $h|_V = 1$.

Now we construct f. Let $p \in \Omega$ and $k \in K$. Seeing that $\partial(s_{k,p}C_k) \subseteq \partial V_p \subseteq R^+$, we have

$$\begin{aligned} \overline{C_k \setminus s_{k,p}^{-1}R^+} &\subseteq \overline{C_k} \setminus s_{k,p}^{-1}R^+ \\ &= s_{k,p}^{-1}(s_{k,p}\overline{C_k} \setminus R^+) \\ &\subseteq s_{k,p}^{-1}(s_{k,p}\overline{C_k} \setminus \partial(s_{k,p}C_k)) = \operatorname{int}(C_k) \subseteq C_k \subseteq V_{1,1}. \end{aligned}$$

Thus we can find a function $f_1 \in C(X, [0,1])$ such that $f_1|_{C_k \setminus s_{k,p}^{-1}R^+} = 1$ for all $p \in \Omega$ and $k \in K$ and $f_1|_{X \setminus V_{1,1}} = 0$. An argument essentially identical to the one given above for h_1 shows that $\{d_p f_1 d_p^*\}_{p\in\Omega}$ is a set of pairwise orthogonal functions in $C(X)$. Moreover, since f_1 is supported on $V_{1,1}$ the function $d_p f_1 d_p^* = \sum_{k\in K} 1_{s_{k,p}C_k}(s_{k,p}f_1)$ is supported on V_p, so that $f = \sum_{p\in\Omega} d_p f_1 d_p^*$ is a function in $C(X)$ which is supported on V, i.e., $f|_R = 0$. Exactly as was done above for h, one can show that for all $p, q \in \Omega$ the function f commutes with every $e_{p,q}$ and the element $fe_{p,q}$ belongs to $C(X) \rtimes_\lambda G$. Observe finally that, for every $x \in V \setminus R^+$, taking $p \in \Omega$ and $k \in K$ such that $x \in s_{k,p}C_k \setminus R^+$ and using that $s_{k,p}^{-1}(x) \in C_k \setminus s_{k,p}^{-1}R^+$ we have

$$f(x) = (d_p f_1 d_p^*)(x) = (s_{k,p}f_1)(x) = 1,$$

that is, $f|_{V\setminus R^+} = 1$.

We are ready to construct z_1. Choose some permutation κ_1 of $\{1, \dots, n^2\}$ that for each $i = 1, \dots, n$ shifts the numbers in the interval $\{(i-1)(n-1)+2, (i-1)(n-1)+3, \dots, (i-1)(n-1)+n\}$ by $i-1$ in the positive direction. Write S for the permutation matrix in M_{n^2} corresponding to κ_1^{-1}, i.e., the matrix whose ℓ-th row for a given $1 \le \ell \le n^2$ is the same as the $\kappa_1(\ell)$-th row of the identity matrix. We could interpret the $(1+n^2) \times (1+n^2)$ block diagonal matrix $\operatorname{diag}(1, S)$ as an element of $B(X) \rtimes_\lambda G$ using the family of Murray–von Neumann equivalences between $1_{V_{1,1}}$ and the other indicator functions $1_{V_{i,j}}$ induced from our equivalences between the sets $V_{i,j}$. But this element would not lie in $C(X) \rtimes_\lambda G$ unless all of the sets at play were clopen, and so it is not directly of use. The unitary z_1 will be a proxy for $\operatorname{diag}(1, S)$ inside the C*-algebra $C(X) \rtimes_\lambda G$ such that the product $z_1 b$ has the following matrix form (in the illustrative case $n = 3$), which is the same form as if we multiplied b on the left by $\operatorname{diag}(1, S)$ (interpreted as above as an element of $B(X) \rtimes_\lambda G$) instead of z_1:

$$\left[\begin{array}{c|ccc|ccc|ccc} 0 & 0 & * & * & 0 & * & * & 0 & * & * \\ \cline{2-10} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ \cline{2-10} 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ \cline{2-10} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unitary group $\mathscr{U}(M_{n^2})$ is path-connected, there is a continuous map $W : [0,1] \to \mathscr{U}(M_{n^2})$ such that $W(0) = 1_{M_{n^2}}$ and $W(1) = S$. Let $\beta : \Omega \to \{1,\dots,n^2\}$ be the bijection $(i,j) \mapsto (i-1)n+j$ giving the lexicographic ordering of Ω. Define $W^\dagger(t) = \sum_{p,q\in\Omega} W(t)_{\beta(p),\beta(q)} e_{p,q}$, for $t \in [0,1]$. Given that $W^\dagger$ is W composed with the isomorphism $M_{n^2} \cong C^*(\{e_{p,q}\}_{p,q\in\Omega})$ that identifies $e_{p,q}$ with the standard matrix unit $e_{\beta(p),\beta(q)}$ in M_{n^2}, we have

$$W^\dagger(t)W^\dagger(t)^* = \sum_{p\in\Omega} e_{p,p} = 1_{\tilde{V}} = W^\dagger(t)^*W^\dagger(t)$$

for all $t \in [0,1]$.

For all $p,q \in \Omega$ define a map $F_{p,q} : [0,1] \to \mathbb{C}$ by $F_{p,q}(t) = W(t)_{\beta(p),\beta(q)}$ (that is, $F_{p,q}(t)$ is the $(\beta(p),\beta(q))$-coordinate of the matrix $W(t)$). Note that $F_{p,q}$ is continuous. Since $f \in C(X)$ is a positive contraction, we may apply the functional calculus to define $g_{p,q} := F_{p,q}(f) \in C(X)$. Since $g_{p,q}$ belongs to $C^*(f,1)$ it commutes with $e_{r,s}$ for all $r,s \in \Omega$, and we have $g_{p,q}(x) = W(f(x))_{\beta(p),\beta(q)}$ for all $x \in X$. Define

$$z_1 = (1-h) + h \cdot \sum_{p,q\in\Omega} g_{p,q}e_{p,q} \in C(X) \rtimes_\lambda G.$$

Since $C(X) \rtimes_\lambda G$ is stably finite, to show that z_1 is unitary it suffices to verify that it is an isometry. This we do as follows, using in the third step the equality $(1-h)1_R = (1-h)$ (which follows from $h|_V = 1$) and an application of Lemma 4.5 (given the unitarity of $\sum_{p,q\in\Omega} g_{p,q}(x)e_{p,q} = W^\dagger(f(x))$ in the C*-subalgebra $C^*(\{e_{p,q}\}_{p,q\in\Omega})$ for all $x \in X$), in the fourth step that $g_{p,q}1_R = \delta_{p,q}1_R$ (since $f|_R = 0$), and in the fifth step that $h1_{\tilde{V}} = h$ and $(1-h)1_R = (1-h)$:

$$\begin{aligned}
z_1^*z_1 &= \Big((1-h) + h \cdot \sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big)^* \Big((1-h) + h \cdot \sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big) \\
&= (1-h)^2 + (1-h)h\Big(\sum_{p,q\in\Omega} g_{p,q}e_{p,q} + \sum_{p,q\in\Omega} \overline{g_{p,q}}e_{q,p}\Big) \\
&\qquad\qquad + h^2\Big(\sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big)^*\Big(\sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big) \\
&= (1-h)^2 + (1-h)h1_R\Big(\sum_{p,q\in\Omega} g_{p,q}e_{p,q} + \sum_{p,q\in\Omega} \overline{g_{p,q}}e_{q,p}\Big) + h^2 1_{\tilde{V}} \\
&= (1-h)^2 + (1-h)h1_R\Big(2 \cdot \sum_{p\in\Omega} e_{p,p}\Big) + h^2 \\
&= (1-h)^2 + 2(1-h)h + h^2 \\
&= 1.
\end{aligned}$$

Next we construct z_2. Choose a permutation κ_2 of $\{1,\dots,n^2\}$ that for each $i = 1,\dots,n$ shifts the numbers in the interval $\{(i-1)n+2, (i-1)n+3, \dots, in\}$ by $n-i$ in the positive direction. Write T for the permutation matrix in M_{n^2} corresponding to κ_2. The unitary z_2 will be a proxy for $\mathrm{diag}(1, T^*)$ inside $C(X) \rtimes_\lambda G$ such that, in the illustrative case $n = 3$, the product z_1bz_2

takes the matrix form

$$\left[\begin{array}{c|ccc|ccc|ccc} 0 & * & * & * & * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

For general n the matrix will have a similar strictly upper triangular form.

Take a continuous map $Z : [0,1] \to \mathscr{U}(M_{n^2})$ such that $Z(0) = 1_{M_{n^2}}$ and $Z(1) = T^*$. As before, for all $p, q \in \Omega$ define $H_{p,q} : [0,1] \to \mathbb{C}$ by $H_{p,q}(t) = Z(t)_{\beta(p),\beta(q)}$ and set $w_{p,q} = H_{p,q}(f) \in C(X)$. Then $w_{p,q}$ commutes with the matrix units $e_{r,s}$ and $w_{p,q}(x) = Z(f(x))_{\beta(p),\beta(q)}$ for all $x \in X$. Set

$$z_2 = (1-h) + h \cdot \sum_{p,q\in\Omega} w_{p,q} e_{p,q} \in C(X) \rtimes_\lambda G.$$

By the same arguments as for z_1, the element z_2 is a unitary.

Let us now finally verify that the matrix representation of $z_1 b z_2$ is strictly upper diagonal. We write $z_1 b z_2 = 1_R z_1 b z_2 + 1_V z_1 b z_2$ and show separately that these two summands take a certain form.

We show that $1_R z_i = 1_R = z_i 1_R$ for $i = 1, 2$. Since $1_R g_{p,q} = \delta_{p,q} 1_R$ for all $p, q \in \Omega$ and $h 1_{\tilde{V}} = h$, we have

$$\begin{aligned} 1_R z_1 &= 1_R(1-h) + h \cdot \sum_{p,q\in\Omega} 1_R g_{p,q} e_{p,q} \\ &= 1_R(1-h) + h \cdot \sum_{p\in\Omega} 1_R e_{p,p} \\ &= (1-h)1_R + h 1_R 1_{\tilde{V}} = 1_R. \end{aligned}$$

Since z_1 is a unitary, it follows that $1_R z_1 = 1_R = z_1 1_R$. A similar argument shows that $1_R z_2 = 1_R = z_2 1_R$. We also observe that z_2 commutes with 1_V given that for all $p, q \in \Omega$ we have

$$\begin{aligned} e_{p,q} 1_V &= d_p d_q^* 1_{\tilde{V}_q} 1_V = d_p d_q^* 1_{V_q} \\ &= \Big(\sum_{k\in K} u_{s_{k,p}} 1_{\tilde{C}_k} u_{s_{k,q}}^* \Big) 1_{V_q} \\ &= \Big(\sum_{k\in K} u_{s_{k,p}} u_{s_{k,q}}^* 1_{s_{k,q}\tilde{C}_k} \Big) 1_{V_q} \\ &= \sum_{k\in K} u_{s_{k,p}} u_{s_{k,q}}^* 1_{s_{k,q}C_k} \\ &= \sum_{k\in K} 1_{s_{k,p}C_k} u_{s_{k,p}} u_{s_{k,q}}^* = 1_{V_p} e_{p,q} = 1_V e_{p,q}. \end{aligned}$$

A similar argument shows that z_1 commutes with 1_V. Since $b1_R = 0$ and hence $b = b1_V$, we thereby obtain

$$1_R z_1 b z_2 = 1_R b z_2 = 1_R b 1_V z_2 = 1_R b z_2 1_V. \tag{4.7}$$

Next we compute $1_V z_1 b z_2$. Given $r = (i,j) \in \Omega$ with $j \neq 1$ we have, applying the fact that $g_{p,q}$ and h commute with the matrix units $e_{p,q}$ and using the equalities $h1_V = 1_V$ and $g_{p,q}1_{V\setminus R^+} = S_{\beta(p),\beta(q)}1_{V\setminus R^+}$,

$$\begin{aligned} z_1 1_{V_r\setminus R^+} &= (1-h)1_R 1_{V_r\setminus R^+} + \Big(h \cdot \sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big)1_{V_r\setminus R^+} \\ &= \Big(h \cdot \sum_{p,q\in\Omega} g_{p,q}e_{p,q}\Big)1_{V_r\setminus R^+} \\ &= \Big(\sum_{p,q\in\Omega} S_{\beta(p),\beta(q)}e_{p,q}\Big)1_{V_r\setminus R^+} \\ &= \Big(\sum_{p,q\in\Omega} S_{\beta(p),\beta(q)}e_{p,q}\Big)1_{\tilde{V}_r}1_{V_r\setminus R^+} \\ &= \Big(\sum_{p,q\in\Omega} S_{\beta(p),\beta(q)}e_{p,q}\Big)e_{r,r}1_{V_r\setminus R^+} \\ &= \Big(\sum_{p\in\Omega} S_{\beta(p),\beta(r)}e_{p,r}\Big)1_{V_r\setminus R^+}. \end{aligned} \tag{4.8}$$

Set $\Upsilon = \{((i,j),(i,j')) \in \Omega\times\Omega : 1 \leq i \leq n,\ 2 \leq j,j' \leq n\}$. Combining with (4.1), it follows that

$$1_V z_1 b = z_1 1_V b = \sum_{p\in\Omega,\,(q,r)\in\Upsilon} S_{\beta(p),\beta(q)}e_{p,q}1_{V_q\setminus R^+} b 1_{V_r\setminus R^+}.$$

Now given $r = (i,j') \in \Omega$ with $j' \neq 1$, a computation similar to (4.8) shows that

$$1_{V_r\setminus R^+} z_2 = \sum_{t\in\Omega} T^*_{\beta(r),\beta(t)}1_{V_r\setminus R^+}e_{r,t} = \sum_{t\in\Omega} T_{\beta(t),\beta(r)}1_{V_r\setminus R^+}e_{r,t},$$

and so combining this with the previous display we obtain

$$1_V z_1 b z_2 = \sum_{p,t\in\Omega,\,(q,r)\in\Upsilon} S_{\beta(p),\beta(q)}e_{p,q}1_{V_q\setminus R^+} b 1_{V_r\setminus R^+} T_{\beta(t),\beta(r)}e_{r,t}.$$

Since $T_{\beta(t),\beta(r)} = 1$ if $\beta(t) = \kappa_2(\beta(r))$ and is 0 otherwise, and $S_{\beta(p),\beta(q)} = 1$ if $\beta(p) = \kappa_1^{-1}(\beta(q))$ and is 0 otherwise, this formula can be rewritten as

$$1_V z_1 b z_2 = \sum_{(q,r)\in\Upsilon} e_{(\beta^{-1}\circ\kappa_1^{-1}\circ\beta)(q),q}1_{V_q\setminus R^+} b 1_{V_r\setminus R^+} e_{r,(\beta^{-1}\circ\kappa_2\circ\beta)(r)}. \tag{4.9}$$

Formulas (4.7) and (4.9), together with the fact that $1_{V_p}e_{p,q} = e_{p,q}1_{V_q}$ for all $p,q \in \Omega$, show us that we can write

$$z_1 b z_2 = 1_R w 1_V + \sum_{(q,r)\in\Upsilon} 1_{V_{(\beta^{-1}\circ\kappa_1^{-1}\circ\beta)(q)}} y_{q,r} 1_{V_{(\beta^{-1}\circ\kappa_2\circ\beta)(r)}}$$

for some elements $w, y_{q,r} \in B(X) \rtimes_\lambda G$. In the matrix representation of z_1bz_2, the term $1_R w 1_V$ accounts for the top row of possibly nonzero terms excluding the top left diagonal entry (which is zero), while the sum over Υ produces a $n^2 \times n^2$ submatrix which is strictly upper diagonal seeing that $\kappa_1^{-1} \circ \beta(q) < \kappa_2 \circ \beta(r)$ for all $(q,r) \in \Upsilon$. Thus the matrix representation of z_1bz_2 is strictly upper diagonal, and so $(z_1bz_2)^{n^2+1} = 0$, which establishes the desired nilpotence. □

Theorem 4.7. *Suppose that G is countably infinite. Let $G \curvearrowright X$ be a topologically free minimal action on the Cantor set with $M_G(X) \neq \emptyset$, and suppose that it is weakly squarely divisible. Then the reduced crossed product $C(X) \rtimes_\lambda G$ has stable rank one.*

Proof. Let a be a non-invertible element of $C(X) \rtimes_\lambda G$. By [53, Proposition 6.2] (which assumes freeness although the proof also works for topologically free actions) we may assume, by rotating a by suitable unitaries and perturbing, that $a1_O = 1_O a = 0$ for some nonempty clopen set $O \subseteq X$ and that a has the form $\sum_{s\in E} g_s u_s$ for some finite symmetric set $e \in E \subseteq G$ and functions $g_s \in C(X)$.

As in the proof of Theorem 4.6, we now aim to show that a can be multiplied on the left and right by suitable unitaries in $C(X) \rtimes_\lambda G$ so as to obtain a nilpotent element.

Since the action is weakly squarely divisible there are pairwise disjoint clopen sets $O_0, O_1, O_2 \subseteq X$ with $O_0 \subseteq O$ and a finite symmetric set $e \in F \subseteq G$ such that $O_1 \sqcup O_2 \prec_F O_0$ and the action is (O_1, O_2, FEF)-squarely divisible. The subequivalence $O_1 \sqcup O_2 \prec_F O_0$ means that we can find a partition of $O_1 \sqcup O_2$ into clopen sets A_s for $s \in F$ such that the sets sA_s for $s \in F$ are pairwise disjoint and contained in O_0. Writing C for the complement of $(O_1 \sqcup O_2) \sqcup \bigsqcup_{s\in F} sA_s$ in X, we can then define, by Remark 4.2, a self-adjoint unitary in $C(X) \rtimes_\lambda G$ by

$$u = 1_C + \sum_{s\in F} u_s 1_{A_s} + \sum_{s\in F} u_{s^{-1}} 1_{sA_s}.$$

Setting $a' = uau^{-1} = uau$ we have $a'1_{O_1\sqcup O_2} = 1_{O_1\sqcup O_2} a' = 0$, and so replacing a by a' we may assume that a has the form $\sum_{s\in FEF} f_s u_s$ for some functions $f_s \in C(X)$ (using the fact that $e \in F = F^{-1}$) and satisfies $a1_{O_1\sqcup O_2} = 1_{O_1\sqcup O_2} a = 0$.

Since the action is (O_1, O_2, FEF)-squarely divisible and FEF contains the support of a as an element of the algebraic crossed product, we can now proceed as in the proof of Theorem 4.6 to find unitaries $z_1, z_2 \in C(X) \rtimes_\lambda G$ such that z_1az_2 is nilpotent (this is the key point where the difference between definitions of square divisibility and weak square divisibility plays out and why we need to work in the Cantor setting here, where we can control the support of a when replacing it with a'). In fact we can construct z_1 and z_2 much more easily than in the proof of Theorem 4.6 by proceeding as follows using the zero-dimensional characterization of (O_1, O_2, FEF)-squarely divisible given by Proposition 3.9, which yields an $n \in \mathbb{N}$ and a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise FEF-disjoint clopen subsets of X such that, defining $V = \bigsqcup_{i,j=1}^n V_{i,j}$, $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = V^c$, and $B = V \cap (V^{FEF})^c$, one has the following:

(i) $V_{i,1} \prec O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ for every $i = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V_1 \cup B)^c$, and
(iii) $B \prec O_2 \cap R$.

In all of the above subequivalences the two clopen sets are disjoint, and so by Remark 4.2 we can construct associated (involutive) unitaries u_i for $i = 1, \dots, n$, v, and w in $C(X) \rtimes_\lambda G$. By

Remark 4.3, the unitaries $u_1, \ldots, u_n$ pairwise commute, and also commute with v. Then there exist clopen sets $Y_{i,1} \subseteq O_1 \cap \bigsqcup_{j=2}^{n} V_{i,j} \cap B^c$ for $i = 1, \ldots, n$, $Y_2 \subseteq O_2 \cap V \cap (V_1 \cup B)^c$, and $Y_3 \subseteq O_2 \cap R$ such that

- $u_i 1_{V_{i,1}} u_i^* = 1_{Y_{i,1}}$ and $u_i 1_C = 1_C u_i = 1_C$ for $C = (V_{i,1} \sqcup Y_{i,1})^c$ with $1 \leq i \leq n$,
- $v 1_R v^* = 1_{Y_2}$ and $v 1_C = 1_C v = 1_C$ for $C = (R \sqcup Y_2)^c$,
- $w 1_B w^* = 1_{Y_3}$ and $w 1_C = 1_C w = 1_C$ for $C = (B \sqcup Y_3)^c$.

Set $u = u_1 \cdots u_n$ and $Y_1 = \bigsqcup_{i=1}^{n} Y_{i,1}$. Then $u 1_{V_1} u^* = 1_{Y_1}$ and $u 1_C = 1_C u = 1_C$ for $C = (V_1 \sqcup Y_1)^c$.

Define $b = u^* w a u v$. By partitioning the unit in $C(X) \rtimes_\lambda G$ as $1_R + \sum_{i,j=1}^{n} 1_{V_{i,j}}$ and multiplying b on either side by the projections in this sum we represent b as an $(n^2+1) \times (n^2+1)$ matrix, which we will now argue takes the form (assuming $n = 3$ for the purpose of illustration)

$$\left[\begin{array}{c|ccc|ccc|ccc}
0 & 0 & * & * & 0 & * & * & 0 & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{array}\right],$$

where the first row and column correspond to the set R and the remaining rows and columns correspond to the sets $V_{i,j}$ as ordered lexicographically with respect to the pairs i, j.

Since Y_2 is disjoint from $V_1 \sqcup Y_1$ and contained in O_2, we have

$$auv1_R = au1_{Y_2}v = a1_{Y_2}v = 0,$$

so that $b1_R = 0$, which accounts for the zero leftmost column in the matrix representation of b.

Next, since V_1 is disjoint from $R \sqcup Y_2$ and Y_1 is a subset of O_1, we also have

$$auv1_{V_1} = au1_{V_1} = a1_{Y_1}u = 0,$$

so that $b1_{V_1} = 0$, which accounts for the other n columns of zeros.

Since Y_1 is disjoint from $B \sqcup Y_3$ and contained in O_1, we have

$$1_{V_1}u^*wa = u^*1_{Y_1}wa = u^*1_{Y_1}a = 0$$

and hence $1_{V_1}b = 0$, which accounts for the n rows of zeros in the lower right $n^2 \times n^2$ submatrix.

Now let $1 \leq i, i' \leq n$ with $i \neq i'$ and let $2 \leq j, j' \leq n$. Writing $p = (i,j)$ and $q = (i',j')$ with V_p and V_q interpreted accordingly as $V_{i,j}$ and $V_{i',j'}$, let us check that $1_{V_p}u^*wauv1_{V_q} = 0$, which will account for the remaining zeros in the matrix representation of b. Since $R \setminus Y_3$ is disjoint from the supports of u and w, Y_1 is contained in O_1, and V_p is disjoint from R, we have

$$1_{V_p}u^*w1_Ra = 1_{V_p}u^*w1_{R \setminus Y_3}a = 1_{V_p}u^*1_{R \setminus Y_3}a = 1_{V_p}1_{R \setminus Y_3}a = 0.$$

On the other hand, since Y_3 is disjoint from the support of u and V_p is disjoint from Y_3 we have

$$1_{V_p}u^*w1_Ba = 1_{V_p}u^*1_{Y_3}wa = 1_{V_p}1_{Y_3}wa = 0.$$

For every $s \in FEF$, using that FEF is symmetric we have $V^{FEF} \subseteq sV$ so that $sR \subseteq B \sqcup R$. It follows that $a1_R = 1_{R\sqcup B}a1_R$ by the representation of a via the set FEF. The above two facts yield

$$1_{V_p}u^*wa1_R = 1_{V_p}u^*w(1_R + 1_B)a1_R = 0.$$

Since v is self-adjoint and R is disjoint from the support of u, it follows that

$$1_{V_p}u^*wauv1_{V_q\cap Y_2} = 1_{V_p}u^*wau1_Rv1_{V_q\cap Y_2} = 1_{V_p}u^*wa1_Rv1_{V_q\cap Y_2} = 0. \tag{4.10}$$

Next, set $Z = \bigsqcup_{k=1}^n V_{i',k}$. By the FEF-disjointness of the sets $V_{i,j}$, we have that $sZ \subseteq R \sqcup Z$ for every $s \in FEF$. The representation of a as a sum indexed by FEF, along with the inclusion $Y_3 \subseteq O_2$, then implies

$$a1_Z = 1_{R\sqcup Z}a1_Z = 1_{(R\setminus Y_3)\sqcup Z}a1_Z,$$

and since the support of w is $B \sqcup Y_3$ we have

$$w1_{(R\setminus Y_3)\sqcup Z} = 1_{R\sqcup Z}w1_{(R\setminus Y_3)\sqcup Z}.$$

Moreover $u^*1_{R\sqcup Z} = 1_{R\sqcup Z}u^*1_{R\sqcup Z}$ by the definition of u and the fact that it is self-adjoint. Putting these facts together, and using the disjointness of V_p and $R \sqcup Z$, we obtain

$$\begin{aligned}1_{V_p}u^*wa1_Z &= 1_{V_p}u^*w1_{(R\setminus Y_3)\sqcup Z}a1_Z\\ &= 1_{V_p}u^*1_{R\sqcup Z}w1_{(R\setminus Y_3)\sqcup Z}a1_Z\\ &= 1_{V_p}1_{R\sqcup Z}u^*1_{R\sqcup Z}w1_{(R\setminus Y_3)\sqcup Z}a1_Z\\ &= 0\end{aligned}$$

and thus, since $V_q \setminus Y_2$ is disjoint from the support of v and $u1_{V_q\setminus Y_2} = 1_Zu1_{V_q\setminus Y_2}$ by the definition of u and the fact that it is self-adjoint,

$$1_{V_p}u^*wauv1_{V_q\setminus Y_2} = 1_{V_p}u^*wau1_{V_q\setminus Y_2} = 1_{V_p}u^*wa1_Zu1_{V_q\setminus Y_2} = 0.$$

Combining with (4.10) then yields

$$1_{V_p}u^*wauv1_{V_q} = 1_{V_p}u^*wauv(1_{V_q\setminus Y_2} + 1_{V_q\cap Y_2}) = 0.$$

We have thus verified that the matrix representation of b has the desired form.

Set $\Omega = \{1,\dots,n\}^2$. By the equivalence of the sets $V_{i,j}$, which as above we also write as V_p where $p = (i,j) \in \Omega$, we can find a clopen partition $\{C_k\}_{k\in K}$ of $V_{1,1}$ and tuples $s_k = (s_{k,p})_{p\in\Omega}$ of elements of G for $k \in K$ such that $V_p = \bigsqcup_{k\in K} s_{k,p}C_k$ for all $p \in \Omega$. For $p, q \in \Omega$ we define the partial isometry

$$e_{p,q} = \sum_{k\in K} 1_{s_{k,p}C_k}u_{s_{k,p}}u_{s_{k,q}}^{-1}1_{s_{k,q}C_k}$$

These define matrix units in $C(X) \rtimes_\lambda G$. The diagonal matrix units $e_{p,p}$ are the indicator functions 1_{V_p}, with the identity matrix in the resulting copy of M_{n^2} in $C(X) \rtimes_\lambda G$ equal to 1_V.

Let $\beta : \Omega \to \{1,\dots,n^2\}$ be the bijection $(i,j) \mapsto (i-1)n+j$ giving the lexicographic ordering of Ω. Choose some permutation κ_1' of $\{1,\dots,n^2\}$ that for each $i = 1,\dots,n$ shifts the numbers in the interval $\{(i-1)(n-1)+2, (i-1)(n-1)+3, \dots, (i-1)(n-1)+n\}$ by $i-1$ in the positive direction. Choose a permutation κ_2' of $\{1,\dots,n^2\}$ that for each $i = 1,\dots,n$ shifts the numbers in the interval $\{(i-1)n+2, (i-1)n+3, \dots, in\}$ by $n-i$ in the positive direction. Define the permutations $\kappa_1 = \beta^{-1} \circ \kappa_1' \circ \beta$ and $\kappa_2 = \beta^{-1} \circ \kappa_2' \circ \beta$ of Ω.

We now define the two "permutation matrices"

$$z_1 = 1_R + \sum_{p\in\Omega} e_{p,\kappa_1(p)} \qquad \text{and} \qquad z_2 = 1_R + \sum_{p\in\Omega} e_{p,\kappa_2(p)}.$$

In the illustrative case $n = 3$, multiplying b by z_1 on the left yields a matrix of the form

$$\left[\begin{array}{c|ccc|ccc|ccc}
0 & 0 & * & * & 0 & * & * & 0 & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

and then multiplying by z_2 on the right converts this into a matrix of the form

$$\left[\begin{array}{c|ccc|ccc|ccc}
0 & * & * & * & * & * & * & * & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

For general n we similarly obtain a strictly upper triangular matrix, as is readily checked, and so $(z_1 b z_2)^{n^2+1} = 0$, yielding the desired nilpotence. □

5. Square divisibility and amenable groups

Our aim here is to establish Theorem 5.7.

Lemma 5.1. *Let E be a nonempty finite subset of G containing e and let $0 < \delta \leq 2$. Let F be an $(E, \delta/(2|E|))$-invariant finite subset of G. Then every set $F_0 \subseteq F$ with $|F_0| \geq (1 - \delta/(4|E|))|F|$ is (E, δ)-invariant.*

Proof. We have $F_0^E = F^E \setminus \bigcup_{s\in E} s^{-1}(F \setminus F_0)$ and so, using the fact that $\delta \leq 2$ to obtain the last inequality below,

$$\begin{aligned}
|F_0^E| \geq |F^E| - |E||F \setminus F_0| &\geq \left(1 - \frac{\delta}{2|E|}\right)|F| - \frac{\delta}{4}|F| \\
&\geq \left(1 - \frac{\delta}{2|E|} - \frac{\delta}{4(1 - \delta/(4|E|))}\right)|F_0| \\
&\geq (1 - \delta)|F_0|.
\end{aligned}$$

□

We need the following lemma to get the E-disjointness in Lemma 5.4.

Lemma 5.2. *Let $e \in E$ be a finite subset of G and $\delta > 0$. Then for every (E^2, δ)-invariant finite set $K \subseteq G$ the set K^E is (E, δ)-invariant.*

Proof. If K is a (E^2, δ)-invariant finite subset of G then

$$|K^E| \geq |(K^E)^E| = |K^{E^2}| \geq (1 - \delta)|K|. \qquad \square$$

Definition 5.3. Let $\mathscr{T}$ be a collection of finite subsets of G. Let $\varepsilon > 0$ and let E be a finite subset of G containing e. An (ε, E)-*tiling* of a finite set $K \subseteq G$ by sets in $\mathscr{T}$ is a collection $\{T_i\}_{i \in I}$ of right translates of sets in $\mathscr{T}$ such that

(i) the sets ET_i for $i \in I$ are pairwise disjoint subsets of K, and
(ii) $|\bigsqcup_{i \in I} T_i| \geq (1 - \varepsilon)|K|$.

We need the following version of the Ornstein–Weiss quasitiling theorem.

Lemma 5.4. *Suppose that G is amenable. Let E be a finite subset of G containing e and let $0 < \varepsilon \leq 1$. Then there is a finite collection $\mathscr{T}$ of (E, ε)-invariant finite subsets of G, a finite set $D \subseteq G$, and a $\beta > 0$ such that for every (D, β)-invariant finite set $K \subseteq G$ there is an (ε, E)-tiling of K by sets in $\mathscr{T}$.*

Proof. Take an $\varepsilon' > 0$ such that $(1 - \varepsilon')^3 \geq 1 - \varepsilon$. Set $\varepsilon'' = \varepsilon'/(2|E^2|)$. By the Ornstein–Weiss quasitiling theorem (see Theorem 6 in [62] and Theorem 4.36 in [45]) there are a finite collection $\mathscr{T}_0$ of (E^2, ε'')-invariant finite subsets of G, a finite set $D \subseteq G$, and a $\beta > 0$ (more precisely, one can take $\beta = \varepsilon''/8$) such that for every (D, β)-invariant finite set $K \subseteq G$ there is a finite collection $\mathscr{S}$ of right translates of members of $\mathscr{T}_0$ satisfying $\bigcup_{S \in \mathscr{S}} S \subseteq K$ and $|\bigcup_{S \in \mathscr{S}} S| \geq (1 - \varepsilon''/2)|K|$ and pairwise disjoint sets $\tilde{S} \subseteq S$ for $S \in \mathscr{S}$ such that $|\tilde{S}| \geq (1 - \varepsilon''/2)|S|$ for every $S \in \mathscr{S}$. Given such K and $\mathscr{S}$, by Lemma 5.1 the sets $\tilde{S}$ for $S \in \mathscr{S}$ are (E^2, ε')-invariant. Then, by Lemma 5.2, for every $S \in \mathscr{S}$ the set $\tilde{S}^E$ satisfies $|\tilde{S}^E| \geq (1 - \varepsilon')|\tilde{S}|$. Using this in the first step and the fact that $|\tilde{S}| \geq (1 - \varepsilon')|S|$ for every $S \in \mathscr{S}$ at the second step, we have

$$\begin{aligned} \Big| \bigsqcup_{s \in \mathscr{S}} \tilde{S}^E \Big| \geq (1 - \varepsilon') \Big| \bigsqcup_{S \in \mathscr{S}} \tilde{S} \Big| &\geq (1 - \varepsilon')^2 \Big| \bigcup_{S \in \mathscr{S}} S \Big| \\ &\geq (1 - \varepsilon')^3 |K| \geq (1 - \varepsilon)|K|. \end{aligned}$$

Thus $\{\tilde{S}^E\}_{s \in \mathscr{S}}$ forms an (ε, E)-tiling of K by sets in $\mathscr{T}$, where

$$\mathscr{T} := \{T^E : T \subseteq T_0 \text{ for some } T_0 \in \mathscr{T}_0 \text{ for which } |T| \geq (1 - \varepsilon''/2)|T_0|\}.$$

It remains to show that $\mathscr{T}$ consists of (E, ε)-invariant sets. Let $T \subseteq T_0$ for some $T_0 \in \mathscr{T}_0$ for which $|T| \geq (1 - \varepsilon''/2)|T_0|$. Since T_0 is (E^2, ε'')-invariant, Lemma 5.1 implies that T is (E^2, ε')-invariant, and in particular (E^2, ε)-invariant. By Lemma 5.2 the set T^E is (E, ε)-invariant. $\square$

Lemma 5.5. *Let $G \curvearrowright X$ be a minimal action and let $\{O_i\}_{i \in I}$ be a finite collection of nonempty open subsets of X. Then there are a finite set $E \subseteq G$ and $0 < \theta \leq 1/2$ such that every Borel tower (S, V) with an $(E, 1/2)$-invariant shape S satisfies*

$$\mu(O_i \cap SV) \geq \theta \mu(SV)$$

for every $i \in I$ and $\mu \in M_G(X)$.

Proof. It suffices to prove the lemma for one nonempty open set $O \subseteq X$, since then for every $i \in I$ we can find a $0 < \theta_i \leq 1/2$ and a finite set $E_i \subseteq G$ satisfying the conditions in the lemma for O_i, in which case we may take $\theta = \min_{i\in I} \theta_i$ and $E = \bigcup_{i\in I} E_i$.

By minimality and the compactness of X there is a finite set $E \subseteq G$ such that $\bigcup_{t\in E} t^{-1}O = X$. Set $\theta = 1/(2|E|)$.

Let (S, V) be a Borel tower whose shape S is $(E, 1/2)$-invariant and let $\mu \in M_G(X)$. For every $s \in S$ set $n(s) = |\{(t, r) \in E \times S^E : tr = s\}|$ and note that $0 \leq n(s) \leq |E|$. We have

$$\begin{aligned}\frac{1}{2}|S|\mu(V) \leq \left|S^E\right|\mu(V) = \sum_{r\in S^E} \mu(rV) &\leq \sum_{r\in S^E}\sum_{t\in E} \mu(rV \cap t^{-1}O)\\ &= \sum_{r\in S^E}\sum_{t\in E} \mu(trV \cap O)\\ &= \sum_{s\in E} n(s)\mu(sV \cap O)\\ &\leq |E|\mu(SV \cap O)\end{aligned}$$

Dividing by $|E|$ we obtain $\mu(SV \cap O) \geq \theta|S|\mu(V) = \theta\mu(SV)$, as desired. □

The following lemma is a consequence of the portmanteau theorem (see [50, Proposition 3.4]).

Lemma 5.6. *Let $G \curvearrowright X$ be an action, $A \subseteq X$ a closed set, and $\delta > 0$. Then there is an open set $U \supseteq A$ such that $\sup_{\mu\in M_G(X)} \mu(U) < \sup_{\mu\in M_G(X)} \mu(A) + \delta$.*

Theorem 5.7. *Suppose that G is infinite and amenable. Let $G \curvearrowright X$ be a minimal action that has the URP and comparison. Let O_1 and O_2 be nonempty open subsets of X and E a finite subset of G containing e. Then the action is (O_1, O_2, E)-squarely divisible. In particular, the action is squarely divisible.*

Proof. By Lemma 5.5 there exist a finite set $F \subseteq G$ and $0 < \theta \leq 1/2$ such that for every Borel tower (S, V) with $(F, 1/2)$-invariant shape S one has, for $i = 1, 2$,

$$\mu(O_i \cap SV) \geq \theta\mu(SV), \tag{5.1}$$

for every $\mu \in M_G(X)$. Let n be an integer greater than $2/\theta + 1$. Set $\varepsilon = \theta/(20n^2)$.

By Lemma 5.4 there are a finite collection $\mathscr{T}$ of $(EF \cup E, \theta\varepsilon/4)$-invariant finite subsets of G, a finite set $D \subseteq G$, and a $\beta > 0$ such that for every (D, β)-invariant finite set $K \subseteq G$ there is an (ε, E)-tiling of K by sets in $\mathscr{T}$.

Since the action has the URP, there exists an open castle $\{(S_k, V_k)\}_{k\in K}$ with (D, β)-invariant shapes such that, writing $R_0 = X \setminus \bigsqcup_{k\in K} S_kV_k$ for the remainder,

$$\sup_{\mu\in M_G(X)} \mu(R_0) < \frac{\theta\varepsilon}{12n^2}. \tag{5.2}$$

We will moreover assume without loss of generality that the sets $\{s\overline{V_k} : s \in S_k, k \in K\}$ are pairwise disjoint (see for example Lemma 6.3 in [27]) and that $e \in S_k$ for all $k \in K$ (if $e \notin S_k$ then we can pick a $t \in S_k$ and replace S_k by S_kt^{-1} and V_k by $t^{-1}V_k$). Since G is infinite, by improving the almost invariance that we require of the shapes we may assume, for every $k \in K$, that

$$\varepsilon|S_k| > n^2|\mathscr{T}|\max_{T\in\mathscr{T}}|T|. \tag{5.3}$$

Using Lemma 5.6 and the inequality (5.2) together with the fact that $\partial R_0 \subseteq R_0$, we can find an open set $U \supseteq \partial R_0$ satisfying

$$\sup_{\mu\in M_G(X)} \mu(\overline{U}) < \frac{\theta\varepsilon}{12n^2}. \tag{5.4}$$

Since $\partial R_0 = \bigsqcup_{k\in K}\bigsqcup_{s\in S_k} s\partial V_k$, by shrinking U appropriately we may assume that it is the union of sets sU_k for $s \in S_k$ and $k \in K$ with pairwise disjoint closures where U_k is some open set containing ∂V_k. Moreover, we may assume that $s\overline{U_k} \cap t\overline{V_\ell} = \emptyset$ whenever $(s,k) \neq (t,\ell)$ for $s \in S_k$, $t \in S_\ell$ and $k,\ell \in K$. Indeed, since $\{s\overline{V_k} : s \in S_k,\ k \in K\}$ is a disjoint collection we can use a separation argument to find open sets $Q_{s,k} \supseteq s\overline{V_k}$ for $s \in S_k$ and $k \in K$ with pairwise disjoint closures, in which case the sets $U_k := \bigcap_{s\in S_k} s^{-1}(U \cap Q_{s,k})$ for $k \in K$ satisfy the required properties. These properties in particular imply that

$$\overline{U}^c \cap sV_k = s(\overline{U}^c \cap V_k) \tag{5.5}$$

for every $s \in S_k$ and $k \in K$ (using that $e \in S_k$ for every $k \in K$).

Let $k \in K$. By our choice of D and β there exist finite sets $C_{k,T} \subseteq G$ for $T \in \mathscr{T}$ such that the sets ETc for $T \in \mathscr{T}$ and $c \in C_{k,T}$ are pairwise disjoint subsets of S_k and $|\bigsqcup_{T\in\mathscr{T}}\bigsqcup_{c\in C_{k,T}} Tc| \geq (1-\varepsilon)|S_k|$, in which case

$$(1-\varepsilon)|S_k| \leq \sum_{T\in\mathscr{T}} |T||C_{k,T}| \leq |S_k|. \tag{5.6}$$

We claim that the inequality $\varepsilon|S_k| > \max_{T\in\mathscr{T}}|T|$ guaranteed by (5.3) permits us to find sets $C'_{k,T} \subseteq C_{k,T}$ for $T \in \mathscr{T}$ satisfying

$$(1-3\varepsilon)|S_k| \leq \sum_{T\in\mathscr{T}} |T||C'_{k,T}| \leq (1-2\varepsilon)|S_k|. \tag{5.7}$$

To see this, first write $\mathscr{T} = \{T_1,\ldots,T_m\}$. Finding sets $C'_{k,T_i} \subseteq C_{k,T_i}$ for $i = 1,\ldots,m$ which satisfy (5.7) is equivalent to finding integers $0 \leq r_i \leq |C_{k,T_i}|$ (specifying the number of elements we will remove from C_{k,T_i}) for $i = 1,\ldots,m$ so that

$$\sum_{i=1}^m |T_i||C_{k,T_i}| - (1-2\varepsilon)|S_k| \leq \sum_{i=1}^m |T_i|r_i \leq \sum_{i=1}^m |T_i||C_{k,T_i}| - (1-3\varepsilon)|S_k|.$$

Set $L_1 = \sum_{i=1}^m |T_i||C_{k,T_i}| - (1-2\varepsilon)|S_k|$ and $L_2 = \sum_{i=1}^m |T_i||C_{k,T_i}| - (1-3\varepsilon)|S_k|$, and note that $L_2 - L_1 = \varepsilon|S_k|$. Define $\mathcal{A} = \left\{\sum_{i=1}^m |T_i|r_i : 0 \leq r_i \leq |C_{k,T_i}|\right\}$, which is a finite set of nonnegative integers. Our goal is to find an element $a \in \mathcal{A}$ with $L_1 \leq a \leq L_2$.

Pick a maximal element $a \in \mathcal{A}$ satisfying $a \leq L_1$. If $a = L_1$, then the proof of the claim is complete. We therefore may assume that $a < L_1$. Then there must exist some $i \in \{1,\ldots,m\}$ with $r_i < |C_{k,T_i}|$. Now $a + |T_i|$ is an element of $\mathcal{A}$ and by the choice of a we have $a + |T_i| > L_1$. Using that $|T_i| < \varepsilon|S_k|$ we also have $a + |T_i| < L_1 + \varepsilon|S_k| = L_2$, which completes the proof of the claim.

Now define

$$Z = \bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} TC_{k,T}V_k \quad\text{and}\quad Z' = \bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} TC'_{k,T}V_k.$$

Then, for $\mu \in M_G(X)$,

$$\mu(Z) = \sum_{k\in K} \sum_{T\in\mathscr{T}} |T||C_{k,T}|\mu(V_k) \quad\text{and}\quad \mu(Z') = \sum_{k\in K} \sum_{T\in\mathscr{T}} |T||C'_{k,T}|\mu(V_k).$$

In view of (5.7) we conclude that

$$(1-\varepsilon)\mu(R_0^c) \le \mu(Z) \le \mu(R_0^c) \quad\text{and}\quad (1-3\varepsilon)\mu(R_0^c) \le \mu(Z') \le (1-2\varepsilon)\mu(R_0^c).$$

Thus

$$\mu(Z\setminus Z') = \mu(Z) - \mu(Z') \ge \varepsilon\mu(R_0^c) \overset{(5.2)}{>} \frac{\varepsilon}{2}. \tag{5.8}$$

For each $k \in K$ and $T \in \mathscr{T}$ choose pairwise disjoint sets $C_{k,T,i,j} \subseteq C'_{k,T}$ of equal cardinality indexed by $i,j = 1,\dots,n$ so that their union has cardinality greater than $|C'_{k,T}| - n^2$ (for instance, take each $C_{k,T,i,j}$ to have cardinality $\lfloor |C'_{k,T}|/n^2 \rfloor$). Then we have

$$\begin{aligned}\Bigg| \bigsqcup_{T\in\mathscr{T}} \bigsqcup_{i,j=1}^{n} TC_{k,T,i,j} \Bigg| &= \sum_{T\in\mathscr{T}} \sum_{i,j=1}^{n} |T||C_{k,T,i,j}| \\ &> \sum_{T\in\mathscr{T}} |T|(|C'_{k,T}| - n^2) \\ &\ge \sum_{T\in\mathscr{T}} |T||C'_{k,T}| - n^2|\mathscr{T}| \max_{T\in\mathscr{T}} |T| \\ &\overset{(5.7),(5.3)}{>} (1-4\varepsilon)|S_k|.\end{aligned} \tag{5.9}$$

For $i,j = 1,\dots,n$ we now define

$$V_{i,j} = \bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} TC_{k,T,i,j}V_k.$$

Note that the sets $V_{i,j}$ are E-disjoint due to the fact that the sets ETc for $T \in \mathscr{T}$ and $c \in C_{k,T}$ are pairwise disjoint and contained in S_k. The fact that for each $k \in K$ and $T \in \mathscr{T}$ the sets $C_{k,T,i,j}$ for $i,j = 1,\dots,n$ have equal cardinality guarantees that the sets $TC_{k,T,i,j}V_k$ for $i,j = 1,\dots,n$ are pairwise equivalent. It follows that the sets $V_{i,j}$ are pairwise equivalent. Write $V = \bigsqcup_{i,j=1}^n V_{i,j}$. Since $\partial V \subseteq \partial R_0 \subseteq U$, it remains to check that the set $\{V_{i,j}\}_{i,j=1}^n$ together with $R := X \setminus V$ and $B := \overline{V} \cap ((V \cap \overline{U}^c)^E)^c$ satisfy the three itemized conditions in the definition of square divisibility.

For $\mu \in M_G(X)$ we have, using the bound $\mu(R_0) \le \varepsilon \le \varepsilon/(1-4\varepsilon)$ guaranteed by (5.2) to get the last inequality,

$$\mu\Bigg(\bigsqcup_{i,j=1}^{n} V_{i,j} \Bigg) \overset{(5.9)}{\ge} (1-4\varepsilon) \sum_{k\in K} |S_k|\mu(V_k) = (1-4\varepsilon)\mu(R_0^c) \ge 1-5\varepsilon. \tag{5.10}$$

The pairwise equivalence of the sets $\{V_{i,j}\}_{i,j=1}^n$ in conjunction with (5.10) yields, for all $i, j = 1, \ldots, n$,

$$\mu(V_{i,j}) \geq \frac{1-5\varepsilon}{n^2} \geq \frac{1}{2n^2}. \tag{5.11}$$

Letting $1 \leq i, j \leq n$ and using (5.11) we observe that

$$\mu(\overline{U}) \overset{(5.4)}{<} \frac{\varepsilon}{2n^2} \leq \varepsilon\mu(V_{i,j}). \tag{5.12}$$

For a set $\Omega \subseteq \{(i,j) : 1 \leq i, j \leq n\}$ we let $V_\Omega = \bigsqcup_{(i,j)\in\Omega} V_{i,j}$. Using that the sets $\{V_{i,j}\}_{i,j=1}^n$ are E-disjoint, it is straightforward to verify that

$$V_\Omega \cap B^c = (V_\Omega \cap \overline{U}^c)^E. \tag{5.13}$$

We also have that

$$\begin{aligned}(V_\Omega \cap \overline{U}^c)^E &\supseteq \bigsqcup_{(i,j)\in\Omega} (V_{i,j} \cap \overline{U}^c)^E \overset{(5.5)}{=} \bigsqcup_{(i,j)\in\Omega} \Big(\bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} TC_{k,T,i,j}(V_k \cap \overline{U}^c)\Big)^E \\ &\supseteq \bigsqcup_{(i,j)\in\Omega} \bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} T^E C_{k,T,i,j}(V_k \cap \overline{U}^c).\end{aligned} \tag{5.14}$$

Together with the $(E, \theta\varepsilon/4)$-invariance of the sets $T \in \mathscr{T}$ and equation (5.5), this yields, for all $\mu \in M_G(X)$,

$$\mu\big((V_\Omega \cap \overline{U}^c)^E\big) \geq (1 - \theta\varepsilon/4)\mu(V_\Omega \cap \overline{U}^c). \tag{5.15}$$

For $i = 1, 2$ we have, using at the second step that for every $T \in \mathscr{T}$ the set T^E is $(F, 1/2)$-invariant (by the (EF, ε)-invariance of T) and at the second to last step that every $T \in \mathscr{T}$ is (E, ε)-invariant,

$$\begin{aligned}\mu(O_i \cap (V_\Omega \cap \overline{U}^c)^E) &\overset{(5.14)}{\geq} \sum_{(i,j)\in\Omega} \sum_{k\in K} \sum_{T\in\mathscr{T}} \sum_{c\in C_{k,T,i,j}} \mu(O_i \cap T^E c(V_k \cap \overline{U}^c)) \\ &\overset{(5.1)}{\geq} \theta \sum_{(i,j)\in\Omega} \sum_{k\in K} \sum_{T\in\mathscr{T}} \sum_{c\in C_{k,T,i,j}} \mu(T^E c(V_k \cap \overline{U}^c)) \\ &= \theta \sum_{(i,j)\in\Omega} \sum_{k\in K} \sum_{T\in\mathscr{T}} |T^E||C_{k,T,i,j}|\mu(V_k \cap \overline{U}^c) \\ &\geq \theta(1-\varepsilon) \sum_{(i,j)\in\Omega} \sum_{k\in K} \sum_{T\in\mathscr{T}} |T||C_{k,T,i,j}|\mu(V_k \cap \overline{U}^c) \\ &\overset{(5.5)}{=} \theta(1-\varepsilon)\mu(V_\Omega \cap \overline{U}^c).\end{aligned} \tag{5.16}$$

Moreover, since $\overline{V_{i,j}} \subseteq \overline{U} \cup (V_{i,j} \cap \overline{U}^c)$ and $\mu(\overline{U}) < \varepsilon/(2n^2) \overset{(5.11)}{\leq} \varepsilon\mu(\overline{V_{i,j}})$ we have

$$\mu(V_{i,j} \cap \overline{U}^c) \geq \mu(\overline{V_{i,j}}) - \mu(\overline{U}) \geq (1-\varepsilon)\mu(\overline{V_{i,j}})$$

and hence

$$\mu(V_\Omega \cap \overline{U}^c) \geq (1-\varepsilon)\mu(\overline{V_\Omega}). \tag{5.17}$$

Next let $1 \leq i \leq n$. Set $W_i = \bigsqcup_{j=2}^n V_{i,j}$ and $Y_i = O_1 \cap W_i \cap B^c \overset{(5.13)}{=} O_1 \cap (W_i \cap \overline{U}^c)^E$. Since $(1-\varepsilon)^2 \geq 1/2$, for $\mu \in M_G(X)$ we then have

$$\mu(Y_i) \overset{(5.16)}{\geq} \theta(1-\varepsilon)\mu(W_i \cap \overline{U}^c) \overset{(5.17)}{\geq} \theta(1-\varepsilon)^2\mu(\overline{W_i}) \geq \frac{\theta}{2}\mu(\overline{W_i}),$$

and thus, since the sets $\overline{V_{i,j}}$ are pairwise disjoint and pairwise Borel equivalent, and $n > 2/\theta + 1$,

$$\mu(\overline{V_{i,1}}) = \frac{1}{n-1}\mu(\overline{W_i}) \leq \frac{2}{\theta(n-1)}\mu(Y_i) < \mu(Y_i).$$

By hypothesis the action has comparison, and so we infer that $\overline{V_{i,1}} \prec Y_i$, for each $i = 1, \dots, n$, which verifies condition (i) in the definition of square divisibility.

We next show that $R \prec O_2 \cap V \cap V_1^c \cap B^c$, which we will again do via comparison. If $\mu \in M_G(X)$ and $\Omega = \{(i,j) : 1 \leq i \leq n, 2 \leq j \leq n\}$, then $V_\Omega = V \cap V_1^c$ and we have, using at the last step below that $\varepsilon = \theta/(20n^2) \leq \theta(n^2-n)/(20n^2)$,

$$\begin{aligned}
\mu(O_2 \cap V \cap V_1^c \cap B^c) &\overset{(5.13)}{=} \mu(O_2 \cap (V_\Omega \cap \overline{U}^c)^E) \\
&\overset{(5.16),(5.17)}{\geq} \theta(1-\varepsilon)^2\mu(V_\Omega) \\
&\geq \frac{\theta}{2}\mu(V_\Omega) \\
&= \frac{\theta}{2}(n^2-n)\mu(V_{i,j}) \\
&\overset{(5.11)}{\geq} \frac{\theta}{4n^2}(n^2-n) \\
&> 5\varepsilon.
\end{aligned}$$

Since $\mu(R) \leq 5\varepsilon$ by (5.10), comparison then yields condition (ii) in the definition of square divisibility.

Finally we verify, again using comparison, that $B \cup (\overline{U} \cap R) \prec O_2 \cap \overline{V \cup U}^c$. Since $\overline{V}$ is equal to the disjoint union of $\overline{V} \cap ((V \cap \overline{U}^c)^E)^c$ and $(V \cap \overline{U}^c)^E$, given $\mu \in M_G(X)$ we have

$$\begin{aligned}
\mu(B) = \mu(\overline{V} \cap ((V \cap \overline{U}^c)^E)^c) &= \mu(\overline{V}) - \mu((V \cap \overline{U}^c)^E) \\
&\overset{(5.15)}{\leq} \mu(\overline{V}) - (1-\theta\varepsilon/4)\mu(V \cap \overline{U}^c) \\
&= (1-\theta\varepsilon/4)(\mu(\overline{V}) - \mu(V \cap \overline{U}^c)) + \frac{\theta\varepsilon}{4}\mu(\overline{V}) \\
&= (1-\theta\varepsilon/4)\mu(\overline{V} \cap \overline{U}) + \frac{\theta\varepsilon}{4}\mu(\overline{V}) \\
&\leq \mu(\overline{U}) + \frac{\theta\varepsilon}{4},
\end{aligned}$$

and therefore $\mu(B \cup (\overline{U} \cap R)) \leq 2\mu(\overline{U}) + \theta\varepsilon/4$. On the other hand,

$$\overline{V \cup U} = \overline{V} \cup \overline{U} \subseteq \Bigg(\bigsqcup_{k \in K} \bigsqcup_{T \in \mathscr{T}} \bigsqcup_{c \in C_{k,T} \setminus C'_{k,T}} TcV_k\Bigg)^c \cup \overline{U},$$

so that, combining with the equation (5.5),

$$\overline{V \cup U}^c \supseteq \Big(\bigsqcup_{k\in K} \bigsqcup_{T\in\mathscr{T}} \bigsqcup_{c\in C_{k,T}\setminus C'_{k,T}} Tc(V_k \cap \overline{U}^c) \Big).$$

We thus have, using the $(F, 1/2)$-invariance of the sets in $\mathscr{T}$ for the second inequality,

$$\begin{aligned}
\mu(O_2 \cap \overline{V \cup U}^c) &\geq \sum_{k\in K} \sum_{T\in\mathscr{T}} \sum_{c\in C_{k,T}\setminus C'_{k,T}} \mu(O_2 \cap Tc(V_k \cap \overline{U}^c)) \\
&\overset{(5.1)}{\geq} \theta \sum_{k\in K} \sum_{T\in\mathscr{T}} \sum_{c\in C_{k,T}\setminus C'_{k,T}} \mu(Tc(V_k \cap \overline{U}^c)) \\
&= \theta \sum_{k\in K} \sum_{T\in\mathscr{T}} |T|(|C_{k,T}| - |C'_{k,T}|)(\mu(V_k) - \mu(V_k \cap \overline{U})) \\
&\geq \theta\mu(Z \setminus Z') - \mu(\overline{U}) \overset{(5.8)}{>} \frac{\theta\varepsilon}{2} - \mu(\overline{U}).
\end{aligned}$$

Since $\mu(\overline{U}) < \theta\varepsilon/12$, we conclude that

$$\mu(B \cup (\overline{U} \cap R)) \leq 2\mu(\overline{U}) + \frac{\theta\varepsilon}{4} \leq \frac{\theta\varepsilon}{2} - \mu(\overline{U}) < \mu(O_2 \cap \overline{V \cup U}^c),$$

in which case comparison gives $B \cup (\overline{U} \cap R) \prec O_2 \cap \overline{V \cup U}^c$, yielding (iii) in the definition of square divisibility.

The last statement in the theorem follows from Proposition 3.4. □

6. Square divisibility and product actions

Theorem 6.1. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be minimal actions with G infinite, and suppose that the first action has the URP and comparison. Then the product action $G \times H \curvearrowright X \times Y$ given by $(g, h)(x, y) = (gx, hy)$ is squarely divisible.*

Proof. Let $O_X \subseteq X$ and $O_Y \subseteq Y$ be nonempty open sets. It is enough to verify that the product action is $O_X \times O_Y$-squarely divisible.

Since $H \curvearrowright Y$ is minimal, for every $y \in Y$ there are an open neighbourhood $V_y \subseteq Y$ of y and an $h \in H$ such that $hV_y \subseteq O_Y$. A compactness argument then shows that there is a finite open cover $\{V_h\}_{h\in F}$ of Y indexed by a finite set $F \subseteq H$ such that $hV_h \subseteq O_Y$ for every $h \in F$.

Since $G \curvearrowright X$ has the URP it is essentially free, i.e., for every $\mu \in M_G(X)$ the action $G \curvearrowright (X, \mu)$ is free. The minimality of $G \curvearrowright X$ then implies that $G \curvearrowright X$ is topologically free. There thus exists an $x \in X$ such that $sx \neq tx$ for all distinct $s, t \in G$. Since X has no isolated points (as ensues from the minimality and topological freeness of the action of G together with the infiniteness of G) and the action of G is minimal, it follows that we can find elements $g_h \in G \setminus \{e\}$ indexed by $h \in F$ such that the points g_hx for $h \in F$ are distinct and contained in O_X. A separation argument then yields an open neighbourhood O_0 of x such that the sets $\{g_hO_0\}_{h\in F}$ are pairwise disjoint subsets of O_X with $\overline{O_0} \cap g_hO_0 = \emptyset$ for every $h \in F$.

Again using the nonexistence of isolated points in X, we can find nonempty open sets $O_1, O_2 \subseteq X$ such that $\overline{O_1} \cap \overline{O_2} = \emptyset$ and $\overline{O_1} \sqcup \overline{O_2} \subseteq O_0$. Set $O'_0 = (O_X \setminus \overline{O_0}) \times O_Y$ and $O'_i = O_i \times Y$ for

$i = 1, 2$. Clearly O'_0, O'_1, and O'_2 are nonempty disjoint open subsets of $X \times Y$ with $\overline{O'_1} \cap \overline{O'_2} = \emptyset$ and $O'_0 \cap (\overline{O'_1} \sqcup \overline{O'_2}) = \emptyset$.

To conclude that the product action is $O_X \times O_Y$-squarely divisible it remains to show that $\overline{O'_1} \sqcup \overline{O'_2} \prec O'_0$ and that $G \times H \curvearrowright X \times Y$ is (O'_1, O'_2, E)-squarely divisible for a given finite set $e \in E \subseteq G \times H$, which we may assume to be a product set $K \times L$. Since

$$\overline{O'_1} \sqcup \overline{O'_2} = (\overline{O_1} \sqcup \overline{O_2}) \times Y \subseteq \bigcup_{h \in F} (O_0 \times V_h)$$

we have

$$\overline{O'_1} \sqcup \overline{O'_2} \prec \bigsqcup_{h \in F} (g_h, h)(O_0 \times V_h) \subseteq (O_X \setminus \overline{O_0}) \times O_Y = O'_0.$$

On the other hand, by Theorem 5.7 the action $G \curvearrowright X$ is (O_1, O_2, K)-squarely divisible, and if we consider all of the sets that appear in conditions (i)–(iii) in Definition 3.3 as witnesses to this (O_1, O_2, K)-square divisibility and take their products with Y, then it is readily checked that these product sets witness the desired (O'_1, O'_2, E)-squarely divisibility of the product action. □

7. A diagonal action machine

Our goal here is to develop a mechanism for constructing diagonal actions of free products satisfying certain properties that will be employed in later sections to various ends and with varying degrees of power, specifically in the proofs of Propositions 8.5 and 8.6, Lemma 9.1 (on the way to Theorem 9.2), and Theorem 10.4. First we carry out a series of lemmas that help us build a machine, namely Proposition 7.9, that outputs diagonal actions that are minimal and weakly mixing (in the case of Theorem 10.4 we will actually only rely separately on some of these lemmas). The specific inputs to this machine that we will need in our applications are furnished by Proposition 7.12 and involve local freeness and (as is relevant to Lemma 9.1) square divisibility.

We will require some tools from ergodic theory for this program. By the notation (Z, ζ) we always mean an atomless standard probability space, of which there is only one up to measure isomorphism. We write $\mathrm{Act}(G, Z, \zeta)$ for the space of all measure-preserving actions $G \curvearrowright (Z, \zeta)$ equipped with the Polish topology that has as a basis the open sets

$$U_{\alpha,\Omega,E,\delta} = \{\beta \in \mathrm{Act}(G, Z, \zeta) : \mu(\beta_s A \Delta \alpha_s A) < \delta \text{ for all } s \in E \text{ and } A \in \Omega\}$$

where α is an element of $\mathrm{Act}(G, Z, \zeta)$, Ω is a finite collection of measurable subsets of Z, E is a finite subset of G, and $\delta > 0$. For definitions and background on the aspects of ergodic theory at play in this and the next section, including weak mixing, the Koopman representation, and disjointness, see [45] and [31]. Although we use the terminology "weak mixing" and "disjoint" in both the measure-dynamical and topological-dynamical senses, the context will always make it clear which is intended.

Recall that two actions $G \curvearrowright X$ and $G \curvearrowright Y$ are *disjoint* if there is no proper nonempty closed subset $Z \subseteq X \times Y$ which is invariant under the diagonal action $s(x, y) = (sx, sy)$ and satisfies $\pi_X(Z) = X$ and $\pi_Y(Z) = Y$ for the canonical projection maps onto X and Y, respectively.

Lemma 7.1. *Suppose that H is a subgroup of G. Let $G \overset{\alpha}{\curvearrowright} X$ be a minimal action and $G \overset{\beta}{\curvearrowright} Y$ an action whose restriction $H \overset{\tilde{\beta}}{\curvearrowright} Y$ is minimal. Suppose there exists a nonempty closed H-invariant set $A \subseteq X$ such that the action $H \curvearrowright A$ restricting α is minimal and disjoint from $\tilde{\beta}$. Then the diagonal action $G \overset{\alpha\times\beta}{\curvearrowright} X \times Y$ is minimal. In particular, α and β are disjoint.*

Proof. For brevity set $\gamma = \alpha \times \beta$. Let $(x, y) \in X \times Y$ and let $U \subseteq X$ and $V \subseteq Y$ be nonempty open sets. It suffices to show the existence of an $r \in G$ such that $\gamma_r(x, y) \in U \times V$.

By the minimality of α there exists an $s \in G$ such that $\alpha_s U \cap A \neq \emptyset$, in which case $\gamma_s(U \times V) \cap (A \times Y) \neq \emptyset$. Since the actions $H \overset{\alpha}{\curvearrowright} A$ and $H \overset{\tilde{\beta}}{\curvearrowright} Y$ are minimal and disjoint, their product $H \curvearrowright A \times Y$ is minimal. It follows, using the compactness of $A \times Y$, that there are finitely many H-translates of the open set $\gamma_s(U \times V)$ which cover $A \times Y$, which means that we can find a finite collection $\{W_t\}_{t\in F}$ of open subsets of $X \times Y$ indexed by some finite set $F \subseteq H$ such that $A \times Y \subseteq \bigcup_{t\in F} W_t$ and $\gamma_t W_t \subseteq \gamma_s(U \times V)$ for every $t \in F$. By the tube lemma and the compactness of A we can then find an open set $O \subseteq X$ such that $A \subseteq O$ and $O \times Y \subseteq \bigcup_{t\in F} W_t$.

Again using the minimality of α, there exists a $k \in G$ such that $\alpha_k x \in O$. Then $\gamma_k(x, y) \in O \times Y$, and so there is a $t_0 \in F$ for which $\gamma_k(x, y) \in W_{t_0}$. It follows that

$$\gamma_{s^{-1}t_0k}(x, y) \in \gamma_{s^{-1}}\gamma_{t_0} W_{t_0} \subseteq \gamma_{s^{-1}}\gamma_s(U \times V) = U \times V,$$

which completes the proof. □

Recall that a *joining* of two p.m.p. actions $G \curvearrowright (Z_1, \zeta_1)$ and $G \curvearrowright (Z_2, \zeta_2)$ is a probability measure ζ on $Z_1 \times Z_2$ which is invariant under the diagonal action $G \curvearrowright Z_1 \times Z_2$, given by $s(z_1, z_2) = (sz_1, sz_2)$, and projects factorwise onto ζ_1 and ζ_2. The p.m.p. actions $G \curvearrowright (Z_1, \zeta_1)$ and $G \curvearrowright (Z_2, \zeta_2)$ are *disjoint* if their only joining is the product measure $\zeta_1 \times \zeta_2$.

Lemma 7.2. *Suppose G is amenable. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be minimal actions such that $M_G(Y)$ contains a unique measure ν. Suppose that there exists an ergodic $\mu \in M_G(X)$ such that the p.m.p. actions $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ are disjoint. Then the diagonal action $G \curvearrowright X \times Y$ is minimal. In particular, the actions $G \curvearrowright X$ and $G \curvearrowright Y$ are disjoint.*

Proof. Let $(x_0, y_0) \in X \times Y$, and let A denote the orbit closure of (x_0, y_0) under $G \curvearrowright X \times Y$. We will show that $A = X \times Y$.

By the pointwise ergodic theorem [76, 78] (see also [56, Theorem 1.2]), if we fix a tempered Følner sequence $(F_n)_{n\in\mathbb{N}}$ for G (temperedness meaning that $|\bigcup_{k<n} F_k^{-1} F_n| \leq C|F_n|$ for all $n \in \mathbb{N}$ where $C > 0$ is a constant, a property that can be arranged by passing to a suitable subsequence of any given Følner sequence), there exists an $x \in X$ such that

$$\int_X f \, d\mu = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} f(s^{-1}x) \tag{7.1}$$

for all $f \in C(X)$ (actually this holds for μ-a.e. $x \in X$). Since $G \curvearrowright X$ is minimal, the projection of A onto X, being G-equivariant, is surjective. Thus there exists a $y \in Y$ such that $(x, y) \in A$.

Since ν is the unique measure in $M_G(Y)$, we have

$$\int_Y f \, d\nu = \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} f(s^{-1}y) \tag{7.2}$$

for all $f \in C(Y)$, a well-known phenomenon that was originally observed by Oxtoby [63] in the case $G = \mathbb{Z}$. Indeed, if this were not true then one could construct an invariant measure $\rho \in M(Y)$ as a weak* cluster point of point mass averages $\nu_n := |F_n|^{-1} \sum_{s\in F_n} \delta_{s^{-1}y}$ for which $\int_Y f\, d\rho$ is bounded away from $\int_Y f\, d\nu$, in which case $\rho \neq \nu$.

Applying the same convergence-to-invariance principle using the Følnerness of the sets F_n, the point mass averages $\lambda_n := |F_n|^{-1} \sum_{s\in F_n} \delta_{s^{-1}(x,y)} \in M(X \times Y)$ have a weak* cluster point $\lambda \in M_G(X \times Y)$. By passing to a subsequence of $(F_n)_{n\in\mathbb{N}}$ and relabelling, we may assume that $\int_{X\times Y} f\, d\lambda = \lim_{n\to\infty} |F_n|^{-1} \sum_{s\in F_n} f(s^{-1}x, s^{-1}y)$ for all $f \in C(X \times Y)$. Since $(x, y) \in A$ and A is closed and G-invariant, each λ_n is supported on A and hence λ is supported on A. By (7.1) and (7.2), λ projects factorwise onto μ and ν hence by the disjointness of $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ we infer that $\lambda = \mu \times \nu$. In particular, $\mu \times \nu$ is supported on A. By the minimality of $G \curvearrowright X$ and $G \curvearrowright Y$, each of μ and ν have full support and hence $A = X \times Y$. □

Recall that two positive bounded Borel measures μ and ν on X are said to be *mutually singular*, written $\mu \perp \nu$, if there exists a Borel set $A \subseteq X$ for which $\mu(X \setminus A) = 0$ and $\nu(A) = 0$.

Let K be a compact group and let μ and ν be positive bounded Borel measures on K. The *convolution* of μ and ν is defined by $(\mu * \nu)(E) = \int_K \mu(Ez^{-1})\, d\nu(z)$ for Borel sets $E \subseteq K$. The convolution of n copies of μ is written μ^{*n}. Note that if $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ and $\nu = \sum_{j=1}^m \beta_j \delta_{y_j}$ are atomic measures then $\mu * \nu = \sum_{i,j} \alpha_i \beta_j \delta_{x_i y_j}$.

The *conjugate* $\overline{\nu}$ of a measure $\nu \in M(\mathbb{T})$ is defined by $\overline{\nu}(B) = \nu(\{z \in \mathbb{T} : \overline{z} \in B\})$ for Borel sets $B \subseteq \mathbb{T}$.

Lemma 7.3. *Let $\mu \in M(\mathbb{T})$. Then the set of all $\nu \in M(\mathbb{T})$ such that $(\nu+\overline{\nu})^{*n} \perp \mu$ for all $n \in \mathbb{N}$ is a dense G_δ.*

Proof. Let $\omega \in M(\mathbb{T})$, let Υ be a finite subset of $C(\mathbb{T})$, and let $\varepsilon > 0$. Let $m \in \mathbb{N}$ and $\varepsilon' > 0$, to be determined. For $k = 1, \ldots, m$ write $I_{k,m}$ for the half-open arc from $e^{2\pi i(k-1)/m}$ (inclusive) to $e^{2\pi ik/m}$ (exclusive). For every $k = 1, \ldots, m$ find $p_k, q_k \in \mathbb{N} \cup \{0\}$ such that $|\omega(I_{k,m}) - p_k/q_k| < \varepsilon'$ and $\sum_{k=1}^m p_k/q_k = 1$. Take a set $F \subseteq \mathbb{T}$, to be further specified below, whose intersection with the arc $I_{k,m}$ for a given $1 \le k \le m$ contains exactly $p_k \cdot \prod_{j\neq k} q_j$ points. Letting $\kappa \in M(\mathbb{T})$ be the uniform probability measure supported on F, one can verify that

(i) $|\omega(I_{k,m}) - \kappa(I_{k,m})| < \varepsilon'$ for every $k = 1, \ldots, m$.

Write C for the (countable) set of all $z \in \mathbb{T}$ such that $\mu(\{z\}) > 0$. We claim that F can be chosen so that, writing $\overline{F} = \{\overline{z} : z \in F\}$,

(ii) the sets $F^a \overline{F}^b$ are disjoint from C for all integers $a, b \ge 0$.

Set $N_k = p_k \cdot \prod_{j\neq k} q_j$. Then the configuration space U for the possible sets F satisfying our original requirement that $|F \cap I_{k,m}| = p_k \cdot \prod_{j\neq k} q_j$ for every $k = 1, \ldots, m$ can be parameterized as $\prod_{k=1}^m (I_{k,m}^{N_k} \setminus \Delta_{k,m}) \subseteq \prod_{k=1}^m \mathbb{T}^{N_k}$, where $\Delta_{k,m} := \{(x_1, \ldots, x_{N_k}) \in \mathbb{T}^{N_k} : x_i = x_j \text{ for some } i \neq j\}$. Note that $\lambda(U) = \prod_{k=1}^m m^{-N_k} > 0$, where λ denotes normalized Lebesgue measure, since each $\Delta_{k,m}$ can be viewed as a subset of a finite union of hyperplanes in $\mathbb{R}^{N_k}$ via the identification of $\mathbb{T}$ with $[0, 1)$ and hence has measure zero. On the other hand, writing $N = N_1 + \ldots + N_m$, the "bad" set is contained in the countable union

$$A := \bigcup_{c\in C} \bigcup_{a,b=0}^{\infty} \bigcup_{\pi\in\{1,\ldots,N\}^a} \bigcup_{\sigma\in\{1,\ldots,N\}^b} A_{c,a,b,\pi,\sigma}$$

where for $\pi = (i_1, \dots, i_a)$ and $\sigma = (j_1, \dots, j_b)$ the set $A_{c,a,b,\pi,\sigma}$ is defined as

$$\left\{(p_1, \dots, p_N) \in \prod_{k=1}^{m} \mathbb{T}^{N_k} : p_{i_1} \cdots p_{i_a} \cdot \overline{p}_{j_1} \cdots \overline{p}_{j_b} = c\right\}.$$

Again identifying $\mathbb{T}$ with $[0,1)$, the set $A_{c,a,b,\pi,\sigma}$ can be viewed as a subset of a finite union of hyperplanes in $\prod_{k=1}^{m} \mathbb{R}^{N_k}$, namely the intersection of $\prod_{k=1}^{m}[0,1)^{N_k}$ with

$$\bigcup_{\ell=-b-1}^{a-1} \left\{(p_1, \dots, p_N) \in \prod_{k=1}^{m} \mathbb{R}^{N_k} : \sum_{k=1}^{a} p_{i_k} - \sum_{k=1}^{b} p_{j_k} = c + \ell\right\},$$

and hence has measure zero. We conclude that $U \setminus A$ has nonzero measure, and so F can be chosen to satisfy condition (ii) above.

By the uniform continuity of functions in $C(\mathbb{T})$, we can take m large enough and ε' small enough so that condition (i) implies that κ belongs to the weak* open neighbourhood

$$W_{\omega,\Upsilon,\varepsilon} := \left\{\nu \in M(\mathbb{T}) : \left| \int_{\mathbb{T}} f \, d\omega - \int_{\mathbb{T}} f \, d\nu \right| < \varepsilon \text{ for every } f \in \Upsilon\right\}. \tag{7.3}$$

of ω.

Now let $\delta > 0$ and $n \in \mathbb{N}$. Define $Q_{\delta,n}$ to be the set of all $\nu \in M(\mathbb{T})$ for which there exists an open set $V \subseteq \mathbb{T}$ such that $\mu(V) < \delta$ and $(\nu + \overline{\nu})^{*j}(\mathbb{T} \setminus V) < \delta$ for every $j = 1, \dots, n$. By the portmanteau theorem, if $\rho_n \to \rho$ weak* inside $M(Z)$ for some compact metrizable space Z then $\limsup_{n\to\infty} \rho_n(Y) \geq \rho(Y)$ for every closed set $Y \subseteq Z$, a fact which, when combined with the weak* continuity of addition and convolution of bounded positive measures on compact groups [39, Theorem 1.2.2], shows that $Q_{\delta,n}$ is open.

We will also verify that $Q_{\delta,n}$ is dense. Since the sets $W_{\omega,\Upsilon,\varepsilon}$ defined in (7.3) form a basis for the weak* topology on $M(\mathbb{T})$, it suffices to show that each of these sets intersects $Q_{\delta,n}$. For given Υ and ε we have our $F \subseteq \mathbb{T}$ and $\kappa \in W_{\omega,\Upsilon,\varepsilon}$ as above. We claim that κ belongs to $Q_{\delta,n}$. By the disjointness condition in (ii) we can find, for every $z \in \bigcup_{j=1}^{n} \bigcup_{k=0}^{j} F^k \overline{F}^{j-k}$, a small enough open arc U_z in $\mathbb{T}$ containing z so that the union $U = \bigcup_{j=1}^{n} \bigcup_{k=0}^{j} \bigcup_{z \in F^k \overline{F}^{j-k}} U_z$ satisfies $\mu(U) < \delta$. On the other hand, for every $j = 1, \dots, n$ the atomic measure $(\kappa + \overline{\kappa})^{*j}$ is supported on $\bigcup_{k=0}^{j} F^k \overline{F}^{j-k}$. Thus $(\kappa + \overline{\kappa})^{*j}(\mathbb{T} \setminus U) = 0$ for all $j = 1, \dots, n$, which shows that $\kappa \in Q_{\delta,n}$ and hence that $W_{\omega,\Upsilon,\varepsilon}$ intersects $Q_{\delta,n}$.

Now $\bigcap_{m=1}^{\infty} Q_{1/m,m}$ is a dense G_δ set, and we claim that it is equal to the set in the statement of the proposition. First we let $\nu \in M(\mathbb{T})$ be such that $(\nu + \overline{\nu})^{*n} \perp \mu$ for all $n \in \mathbb{N}$ and check that $\nu \in \bigcap_{m=1}^{\infty} Q_{1/m,m}$. Given $m \in \mathbb{N}$, we want to find an open set $V \subseteq \mathbb{T}$ such that $\mu(V) < 1/m$ and $(\nu + \overline{\nu})^{*j}(\mathbb{T} \setminus V) < 1/m$ for every $j = 1, \dots, m$. Since $\sum_{j=1}^{m} (\nu + \overline{\nu})^{*j} \perp \mu$ from our assumption, there is a Borel set $B \subseteq \mathbb{T}$ such that $\mu(B) = 0$ and $(\nu + \overline{\nu})^{*j}(\mathbb{T} \setminus B) = 0$ for every $j = 1, \dots, m$. It follows by the regularity of μ that there is an open set $V \subseteq \mathbb{T}$ satisfying $V \supseteq B$ and $\mu(V) < 1/m$. Then $(\nu + \overline{\nu})^{*j}(\mathbb{T} \setminus V) = 0$ for every $j = 1, \dots, m$ and so ν belongs to $\bigcap_{m=1}^{\infty} Q_{1/m,m}$.

For the reverse inclusion, let $\nu \in \bigcap_{m=1}^{\infty} Q_{1/m,m}$. For every $m \in \mathbb{N}$ we can find an open set $V_m \subseteq \mathbb{T}$ for which $\mu(V_m) < 1/m$ and $(\nu + \overline{\nu})^{*j}(\mathbb{T} \setminus V_m) < 1/m$ for every $j = 1, \dots, m$. Consider the Borel set $V = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} V_{m^2}$. We have $\mu(V) \leq \sum_{m=k}^{\infty} \mu(V_{m^2}) < \sum_{m=k}^{\infty} m^{-2}$ for every $k \in \mathbb{N}$, and hence $\mu(V) = 0$. Finally, writing $\mathbb{T} \setminus V = \bigcup_{k=1}^{\infty} W_k$ where $W_k = \bigcap_{m=k}^{\infty} \mathbb{T} \setminus V_{m^2}$, we have

$(\nu+\overline{\nu})^{*n}(W_k) \leq (\nu+\overline{\nu})^{*n}(\mathbb{T}\setminus V_{m^2}) < 1/m^2$ for all $m \geq \max\{k,\sqrt{n}\}$. Hence $(\nu+\overline{\nu})^{*n}(W_k) = 0$ for all $k \in \mathbb{N}$, which yields $(\nu+\overline{\nu})^{*n}(\mathbb{T}\setminus V) = 0$. Thus $(\nu+\overline{\nu})^{*n} \perp \mu$ for all $n \in \mathbb{N}$. □

Notation 7.4. Given an action $\alpha \in \mathrm{Act}(G,X)$, for a pair of open sets $U_1, U_2 \subseteq X$ we write $N_\alpha(U_1,U_2)$ for the set of all $s \in G$ such that $\alpha_s U_1 \cap U_2 \neq \emptyset$. Given an action $\alpha \in \mathrm{Act}(G,Z,\zeta)$, for a pair of measurable sets $U, V \subseteq Z$ and $\varepsilon > 0$ we write $N_{\alpha,\varepsilon}(U,V)$ for the set of all $s \in G$ such that $\zeta(\alpha_s U \cap V) > \zeta(U)\zeta(V) - \varepsilon$.

For convenient reference we record the following well-known fact. One applies the Ornstein–Weiss quasitower theorem (see [45, Theorem 4.45]) to deduce that for infinite amenable G the conjugacy class of any free action in $\mathrm{Act}(G,Z,\zeta)$ is dense in the set of all free actions in $\mathrm{Act}(G,Z,\zeta)$, which in turn is dense in $\mathrm{Act}(G,Z,\zeta)$ by a result of Glasner and King [32, Appendix (2)].

Lemma 7.5. *Suppose that G is infinite and amenable. Then the conjugacy class of every free action in $\mathrm{Act}(G,Z,\zeta)$ is dense.*

By definition, an action $\alpha \in \mathrm{Act}(G,X)$ is weakly mixing if and only if the intersection of $N_\alpha(U_1,U_2)$ and $N_\alpha(V_1,V_2)$ is nonempty for all nonempty open sets $U_1,U_2,V_1,V_2 \subseteq X$.

Lemma 7.6. *Suppose G is infinite and amenable. Let L be an infinite subset of G. Let Γ be the set of all $\alpha \in \mathrm{Act}(G,Z,\zeta)$ such that $N_{\alpha,\varepsilon}(A_1,A_2) \cap N_{\alpha,\varepsilon}(B_1,B_2) \cap L \neq \emptyset$ for all measurable sets $A_1,A_2,B_1,B_2 \subseteq Z$ and $\varepsilon > 0$. Then Γ contains a dense G_δ subset of $\mathrm{Act}(G,Z,\zeta)$.*

Proof. As explained in the proof of Theorem 1.7 in [45], we can find a countable collection $\mathscr{D}$ of measurable subsets of Z so that for every $\eta > 0$ and every measurable set $A \subseteq Z$ there exists $\tilde{A} \in \mathscr{D}$ with $\zeta(A\Delta\tilde{A}) < \eta$. For a tuple $T = (A_1,A_2,B_1,B_2) \in \mathscr{D}^4$ and $\varepsilon > 0$ write $\mathscr{W}_{T,\varepsilon}$ for the set of all $\alpha \in \mathrm{Act}(G,Z,\zeta)$ such that $N_{\alpha,\varepsilon}(A_1,A_2) \cap N_{\alpha,\varepsilon}(B_1,B_2) \cap L \neq \emptyset$. Let $\alpha \in \mathscr{W}_{T,\varepsilon}$ and choose an $s \in N_{\alpha,\varepsilon}(A_1,A_2) \cap N_{\alpha,\varepsilon}(B_1,B_2) \cap L$. It is straightforward to verify that there exists a sufficiently small $\varepsilon' > 0$ so that whenever $\beta \in U_{\alpha,\{A_1,A_2,B_1,B_2\},s,\varepsilon'}$, then β also belongs to $\mathscr{W}_{T,\varepsilon}$, and so $\mathscr{W}_{T,\varepsilon}$ is open.

There exist free mixing actions in $\mathrm{Act}(G,Z,\zeta)$, e.g., the Bernoulli action $G \curvearrowright (Y^G,\nu^G)$ for some nontrivial standard probability space (Y,ν) (see Section 2.3.1 of [45]), identifying (Z,ζ) with (Y^G,ν^G). Now since the conjugacy class of any free action in $\mathrm{Act}(G,Z,\zeta)$ is dense by Lemma 7.5, the set of mixing actions in $\mathrm{Act}(G,Z,\zeta)$ is dense. Since L is infinite, every mixing action in $\mathrm{Act}(G,Z,\zeta)$ is contained in $\mathscr{W}_{T,\varepsilon}$, and so $\mathscr{W}_{T,\varepsilon}$ is dense in $\mathrm{Act}(G,Z,\zeta)$. It follows that the set $\mathscr{W} := \bigcap_{T\in\mathscr{D}^4}\bigcap_{n=1}^\infty \mathscr{W}_{T,1/n}$ is a dense G_δ.

To complete the proof we will verify that $\mathscr{W} \subseteq \Gamma$. Let $\alpha \in \mathscr{W}$ and $\varepsilon > 0$, and let A_1,A_2,B_1,B_2 be measurable subsets of Z. By the density of $\mathscr{D}$ we can find $\tilde{A}_1,\tilde{A}_2,\tilde{B}_1,\tilde{B}_2 \in \mathscr{D}$ such that $\zeta(\tilde{A}_1\Delta A_1)$, $\zeta(\tilde{A}_2\Delta A_2)$, $\zeta(\tilde{B}_1\Delta B_1)$, and $\zeta(\tilde{B}_2\Delta B_2)$ are all less than $\varepsilon/8$. By the definition of $\mathscr{W}$ there exists an $s \in N_{\alpha,\varepsilon/2}(\tilde{A}_1,\tilde{A}_2) \cap N_{\alpha,\varepsilon/2}(\tilde{B}_1,\tilde{B}_2) \cap L$. We then have

$$\begin{aligned}|\zeta(A_1)\zeta(A_2) - \zeta(\tilde{A}_1)\zeta(\tilde{A}_2)| &\leq |\zeta(A_1) - \zeta(\tilde{A}_1)|\zeta(A_2) + \zeta(\tilde{A}_1)|\zeta(A_2) - \zeta(\tilde{A}_2)| \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}\end{aligned}$$

and hence, using the fact that $A \cap B \subseteq (C \cap D) \cup (A\Delta C) \cup (B\Delta D)$ for any sets A,B,C,D,

$$\zeta(\alpha_s A_1 \cap A_2) \geq \zeta(\alpha_s \tilde{A}_1 \cap \tilde{A}_2) - \zeta(\alpha_s(A_1\Delta\tilde{A}_1)) - \zeta(A_2\Delta\tilde{A}_2)$$

$$> \left(\zeta(\tilde{A}_1)\zeta(\tilde{A}_2) - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{8} - \frac{\varepsilon}{8}$$
$$\geq \zeta(A_1)\zeta(A_2) - \varepsilon.$$

Similarly,

$$\zeta(\alpha_s B_1 \cap B_2) > \zeta(B_1)\zeta(B_2) - \varepsilon.$$

Thus $s \in N_{\alpha,\varepsilon}(A_1, A_2) \cap N_{\alpha,\varepsilon}(B_1, B_2) \cap L$, showing that $\alpha \in \Gamma$. □

Lemma 7.7. *Let $G \overset{\alpha}{\curvearrowright} X$ be an action for which $M_\alpha(X)$ contains a measure of full support. Let H be an infinite subgroup of G. Let $U_1, U_2, V_1, V_2 \subseteq X$ be nonempty open sets and let $s \in N_\alpha(U_1, U_2) \cap N_\alpha(V_1, V_2)$. Then the set $Hs \cap N_\alpha(U_1, U_2) \cap N_\alpha(V_1, V_2)$ is infinite.*

Proof. Let $\mu \in M_\alpha(X)$ be of full support. Then the product measure $\mu \times \mu$ on $X \times X$ has full support. By assumption the open sets $\alpha_s U_1 \cap U_2$ and $\alpha_s V_1 \cap V_2$ are nonempty, and so by Poincaré recurrence (see for example [45, Theorem 2.10]) applied to the diagonal action $H \curvearrowright (X \times X, \mu \times \mu)$ we deduce that the set of all $h \in H$ such that

$$(\alpha_h(\alpha_s U_1 \cap U_2) \times \alpha_h(\alpha_s V_1 \cap V_2)) \cap ((\alpha_s U_1 \cap U_2) \times (\alpha_s V_1 \cap V_2)) \neq \emptyset$$

is infinite. For such h we have in particular $\alpha_{hs} U_1 \cap U_2 \neq \emptyset$ and $\alpha_{hs} V_1 \cap V_2 \neq \emptyset$, i.e., $hs \in N_\alpha(U_1, U_2) \cap N_\alpha(V_1, V_2)$. □

A unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ is *weakly mixing* if it has no nonzero finite-dimensional invariant subspaces, or, equivalently, if for every finite set $\Omega \subseteq \mathcal{H}$ and $\varepsilon > 0$ there exists $s \in G$ such that $|\langle \pi(s)\xi, \zeta\rangle| < \varepsilon$ for all $\xi, \zeta \in \Omega$ (see [45, Theorem 2.23]). For a measure $\mu \in M(\mathbb{T})$, the unitary representation of $\mathbb{Z}$ on $L^2(\mathbb{T}, \mu)$ by the multiplication operators z^n is weakly mixing if and only μ is atomless, as is well known and easy to see.

A unitary operator $U \in \mathcal{B}(\mathcal{H})$ is said to be *weakly mixing* if the associated unitary representation $\mathbb{Z} \to \mathcal{B}(\mathcal{H})$ given by $n \mapsto U^n$ is weakly mixing. This is equivalent to the nonexistence of eigenvectors, i.e., the nonexistence of nonzero $\xi \in \mathcal{H}$ such that $U\xi = \lambda\xi$ for some $\lambda \in \mathbb{C}$.

We additionally require some facts from harmonic analysis which we summarize here. Let $\mathcal{H}$ be a separable Hilbert space. Given a unitary operator $U \in \mathcal{B}(\mathcal{H})$, its *spectral measures* are finite Borel measures $\{\sigma_\xi\}_{\xi \in \mathcal{H}}$ on $\mathbb{T}$ with the property that $\langle U^n \xi, \xi\rangle = \int_{\mathbb{T}} z^n d\sigma_\xi(z)$, for all $n \in \mathbb{Z}$. Moreover, there exists a $\sigma_U \in M(\mathbb{T})$, uniquely determined up to membership in the same measure class, with the property that $\sigma_\xi \ll \sigma_U$ for all $\xi \in \mathcal{H}$ and for every finite Borel measure ν on $\mathbb{T}$ with $\nu \ll \sigma_U$ there exists $\xi \in \mathcal{H}$ for which $\nu \sim \sigma_\xi$. This measure, or more accurately its measure class, is called the *maximal spectral type* of U. For $\nu \in M(\mathbb{T})$, the multiplication operator $M_z \in \mathcal{B}(L^2(\mathbb{T}, \nu))$, given by $f \mapsto zf$, has ν as its maximal spectral type. The maximal spectral type of a tensor product $U_1 \otimes \cdots \otimes U_n$ of unitary operators is the convolution measure $\sigma_{U_1} * \cdots * \sigma_{U_n}$. Finally, the maximal spectral type of a countable direct sum $\bigoplus_{n=1}^\infty U_n$ of unitary operators is given by $\sum_{n=1}^\infty c_n \sigma_{U_n}$, where c_n are any positive numbers satisfying $\sum_{n=1}^\infty c_n = 1$.

For a transformation $T \in \mathrm{Aut}(Z, \zeta)$ we consider its *reduced maximal spectral type* $\sigma_{T,0} \in M(\mathbb{T})$, defined as the maximal spectral type of the unitary operator $U_{T,0} \in \mathcal{B}(L^2_0(Z, \zeta))$ that we obtain by restricting the Koopman operator $U_T \xi = \xi \circ T^{-1}$ on $L^2(Z, \zeta)$ to the closed invariant subspace $L^2_0(Z, \zeta) = L^2(Z, \zeta) \ominus \mathbb{C}1$. Given $\gamma \in \mathrm{Act}(\mathbb{Z}, Z, \zeta)$ we denote by $\sigma_{\gamma,0} \in M(\mathbb{T})$ the reduced maximal spectral type of the generating transformation. Note also that $\sigma_{T^{-1},0} = \overline{\sigma_{T,0}}$ (and the same holds for the "usual", i.e., non-reduced maximal spectral type). Two transformations

$T, S \in \mathrm{Aut}(Z, \zeta)$ are *spectrally disjoint* if $\sigma_{T,0} \perp \sigma_{S,0}$, a property known to imply disjointness (e.g., see [31, Theorem 6.28]).

Let $T_1 \in \mathrm{Aut}(Z_1, \zeta_1)$ and $T_2 \in \mathrm{Aut}(Z_2, \zeta_2)$ be two transformations and consider the associated Koopman operators $U_{T_1} \in \mathcal{B}(L^2(Z_1, \zeta_1))$ and $U_{T_2} \in \mathcal{B}(L^2(Z_2, \zeta_2))$. Using that unitarily equivalent unitary operators have the same spectral measure, one checks that the reduced maximal spectral type $\sigma_{T_1 \times T_2, 0}$ is equivalent to $\sigma_{T_1,0} + \sigma_{T_2,0} + (\sigma_{T_1,0} * \sigma_{T_2,0})$. More generally, the reduced maximal spectral type of a (possibly infinite) product is equivalent to the (normalized) sum of all possible convolutions of reduced maximal spectral types of the factors in the product.

For a unitary operator $U \in \mathcal{B}(\mathcal{H})$, we denote by $\overline{U}$ the induced unitary operator on the conjugate Hilbert space $\overline{\mathcal{H}}$. For $\nu \in M(\mathbb{T})$ we have a unitary isomorphism from $\overline{L^2(\mathbb{T}, \nu)}$ to $L^2(\mathbb{T}, \overline{\nu})$ given by $f \mapsto [z \mapsto \overline{f(\overline{z})}]$, which sends $\overline{M_z}$ to M_z.

Finally, we recall the construction of symmetric tensor products. Let $\mathcal{H}$ be a Hilbert space, let $n \in \mathbb{N}$, and let $\sigma \in \mathrm{Sym}(n)$. Write $U_\sigma \in \mathcal{B}(\mathcal{H}^{\otimes n})$ for the unitary operator determined by $U_\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$ on elementary tensors. The *n-fold symmetric tensor product* of $\mathcal{H}$ is the closed Hilbert subspace

$$\mathcal{H}^{\odot n} := \{\xi \in \mathcal{H}^{\otimes n} : U_\sigma(\xi) = \xi \text{ for all } \sigma \in \mathrm{Sym}(n)\}.$$

Every unitary operator $U \in \mathcal{B}(\mathcal{H})$ determines a unitary operator $U^{\odot n} \in \mathcal{B}(\mathcal{H}^{\odot n})$ via the restriction of $U^{\otimes n}$ to $\mathcal{H}^{\odot n}$.

Lemma 7.8. *Suppose that G is amenable and contains a normal infinite cyclic subgroup. Let Ω be a finite subset of* $\mathrm{Act}(G, Z, \zeta)$*. Then the set of all free weakly mixing actions in $\Omega^\perp$ (i.e., those that are disjoint from all actions in Ω) is a dense G_δ subset of* $\mathrm{Act}(G, Z, \zeta)$*.*

Proof. By a result of [17] (which is stated and proved there for ergodic actions of $G = \mathbb{Z}$, although the argument works more generally) the set $\{\alpha\}^\perp$ is a G_δ for every $\alpha \in \Omega$. The last paragraph in the proof of Theorem 5.21 in [45] shows that the set of all weakly mixing actions in $\mathrm{Act}(G, Z, \zeta)$ is a G_δ, while [32, Appendix (2)] shows that set of all free actions in $\mathrm{Act}(G, Z, \zeta)$ is a G_δ. The intersection of all of these G_δ sets is again a G_δ set, and so we conclude that the set of all free weakly mixing actions in $\Omega^\perp$ is a G_δ subset of $\mathrm{Act}(G, Z, \zeta)$. We note that the set of all free weakly mixing actions in $\mathrm{Act}(G, Z, \zeta)$ is actually a dense G_δ, as can be seen by Lemma 7.5 and the existence of Bernoulli actions, which are weakly mixing, or by applying the general result in [49]. It would thus now be sufficient to show that $\Omega^\perp$ contains at least one free action, since again using Lemma 7.5 we could then conclude that $\Omega^\perp$ is dense. However, the disjointness argument below is already contingent on weak mixing via the atomlessness of ν, and so we will get weak mixing anyway, in addition to freeness.

By assumption G has a normal infinite cyclic subgroup H, which we identify with $\mathbb{Z}$. First we show the existence of a free weakly mixing action in $\Omega^\perp$. For $\alpha \in \Omega$, let $\sigma_{\alpha',0} \in M(\mathbb{T})$ denote the reduced maximal spectral type of the restriction α' of α to H. As shown in the appendix of [32], a generic measure in $M(\mathbb{T})$ is atomless, and so by Lemma 7.3 there is an atomless measure $\nu \in M(\mathbb{T})$ such that $(\nu + \overline{\nu})^{*n} \perp \sigma_{\alpha',0}$ for all $\alpha \in \Omega$ and $n \in \mathbb{N}$. Let $\pi : \mathbb{Z} \to \mathcal{B}(L^2(\mathbb{T}, \nu))$ be the unitary representation associated to the multiplication operator $M_z \in \mathcal{B}(L^2(\mathbb{T}, \nu))$ and let $\mathbb{Z} \overset{\gamma}{\curvearrowright} (M, \mu)$ denote its associated Gaussian action, in which case (M, μ) is standard and atomless (see Appendix E of [45]). Since the measure ν is atomless the representation π is weakly mixing, which implies that the action γ is weakly mixing [45, Theorem 2.38]. The transformation obtained by restricting γ to the generator $1 \in \mathbb{Z}$ gives rise to a Koopman operator $U_\gamma \in \mathcal{B}(L^2(M, \mu))$ along with its restriction $U_{\gamma,0} \in \mathcal{B}(L^2_0(M,\mu))$. As shown in Theorem E.19 in [45], there exists a unitary isomorphism $L^2(M,\mu) \simeq \bigoplus_{n=0}^\infty (L^2(\mathbb{T},\nu) \oplus \overline{L^2(\mathbb{T},\nu)})^{\odot n}$ (with the direct summand for $n = 0$ equal to the complex numbers) which carries the unitary operator U_γ to the unitary operator $\bigoplus_{n=0}^\infty (M_z \oplus \overline{M_z})^{\odot n}$. Since γ is weakly mixing, $U_{\gamma,0}$ is weakly mixing (see [45, Proposition 2.7]), so that $L^2_0(M,\mu)$ is exactly the orthogonal complement of vectors fixed by U_γ. Since ν is atomless, the unitary operators $(M_z \oplus \overline{M_z})^{\otimes n}$ for $n \in \mathbb{N}$ are weakly mixing (see [45, Theorem 2.23]), and so likewise $\bigoplus_{n=1}^\infty (L^2(\mathbb{T},\nu) \oplus \overline{L^2(\mathbb{T},\nu)})^{\odot n}$ is the orthogonal complement of vectors fixed by $\bigoplus_{n=0}^\infty (M_z \oplus \overline{M_z})^{\odot n}$. It follows that the isomorphism $L^2(M,\mu) \simeq \bigoplus_{n=0}^\infty (L^2(\mathbb{T},\nu) \oplus \overline{L^2(\mathbb{T},\nu)})^{\odot n}$ restricts to a unitary isomorphism $L^2_0(M,\mu) \simeq \bigoplus_{n=1}^\infty (L^2(\mathbb{T},\nu) \oplus \overline{L^2(\mathbb{T},\nu)})^{\odot n}$. Consequently, $U_{\gamma,0}$ and $\bigoplus_{n=1}^\infty (M_z \oplus \overline{M_z})^{\odot n}$ are unitarily equivalent via this isomorphism.

For each $n \in \mathbb{N}$, the unitary operator $(M_z \oplus \overline{M_z})^{\odot n}$ is the restriction of the unitary operator $(M_z \oplus \overline{M_z})^{\otimes n}$ to an invariant subspace. Hence, the maximal spectral type of the former is absolutely continuous with respect to the maximal spectral type of $(M_z \oplus \overline{M_z})^{\otimes n}$, which, by the discussion preceding this lemma, is exactly $(\nu + \overline{\nu})^{*n}$. Combining these facts, we conclude that the reduced maximal spectral type $\sigma_{\gamma,0} \in M(\mathbb{T})$ of γ is absolutely continuous with respect to $\sum_{n=1}^\infty 2^{-n}(\nu + \overline{\nu})^{*n}$.

Suppose for a moment that G is equal to the subgroup $H \cong \mathbb{Z}$. Then $\mathbb{Z} \overset{\gamma}{\curvearrowright} (M,\mu)$ (identifying $(M,\mu) \simeq (Z,\zeta)$ via a Borel isomorphism) would be the desired free weakly mixing action in $\Omega^\perp$. Indeed, since $\mathbb{Z} \overset{\gamma}{\curvearrowright} (M,\mu)$ is an ergodic action of $\mathbb{Z}$ it must be free, and since $\sigma_{\gamma,0} \perp \sigma_{\alpha,0}$ for all $\alpha \in \Omega$ it follows from the discussion before this lemma that γ is disjoint from each $\alpha \in \Omega$.

Returning to the general case, our aim is to coinduce γ to an action of G that preserves the desired properties. Let $h \in G$ denote a generator for H. Take a transversal $e \in T \subseteq G$ for G/H, i.e., a set such that $G = \bigsqcup_{t \in T} tH$, and consider the action $G \overset{\rho}{\curvearrowright} (M^T, \mu^T)$ coinduced from γ, defined for $s \in G$ and $t \in T$ and $x \in M^T$ by $(\rho_s x)(t) = \gamma_{k^{-1}}(x(r))$ where k and r are the unique elements of H and T, respectively, that satisfy $rk = s^{-1}t$.

Since γ is free and weakly mixing, so is ρ (see Lemmas 2.1 and 2.2 in [41], respectively). Note that since $H = \langle h \rangle$ is normal in G, we have $(\rho_h x)(t) = \gamma_{t^{-1}ht}(x(t))$ for $x \in M^T$ and $t \in T$. Since the conjugation actions $\mathrm{Ad}\, t : H \to H$ for $t \in G$ are automorphisms, they must map the generator h to h or to h^{-1}. Hence the restriction ρ' of the coinduced action ρ to H is generated by the diagonal product automorphism of (M^T, μ^T) given by either γ_h or $\gamma_{h^{-1}}$ at each coordinate. Given the facts summarized before the lemma, it follows that the reduced maximal spectral type $\sigma_{\rho',0}$ is absolutely continuous with respect to the (countable) infinite normalized sum of all possible convolutions of $\sigma_{\gamma,0}$ and $\overline{\sigma_{\gamma,0}}$. Write $\eta = \sum_{n=1}^\infty 2^{-n}(\nu + \bar{\nu})^{*n}$. Then $\bar{\eta} = \eta$ (since the convolution operation commutes with the conjugation operation for measures, as follows directly from the definitions) and $\eta^{*m} \ll \eta$ for all $m \in \mathbb{N}$. Combining these observations together with the fact that $\sigma_{\gamma,0} \ll \eta$, we deduce that $\sigma_{\rho',0} \ll \eta$, i.e., $\sigma_{\rho',0} \ll \sum_{n=1}^\infty 2^{-n}(\nu + \bar{\nu})^{*n}$.

We have thus shown that for every $\alpha \in \Omega$ the reduced spectral types $\sigma_{\alpha',0}$ and $\sigma_{\rho',0}$ are mutually singular. Therefore the actions α' and ρ' are (spectrally) disjoint (note that α' need not be ergodic, and thus $\sigma_{\alpha',0}$ can have an atom at 1, but since ρ' is ergodic, being a product of weakly mixing actions, its reduced maximal spectral type $\sigma_{\rho',0}$ does not have an atom at 1). The disjointness of α' and ρ' implies that α and ρ are themselves disjoint (we identify ρ as an action on (Z,ζ) using a Borel isomorphism $(M^T,\mu^T) \simeq (Z,\zeta)$). Since ρ is free, by Lemma 7.5 its conjugacy class is dense in $\mathrm{Act}(G,Z,\zeta)$. Since weak mixing and disjointness are preserved under conjugacy, we conclude that the free weakly mixing actions which belong to $\Omega^\perp$ form a dense G_δ subset of $\mathrm{Act}(G, Z, \zeta)$. □

Proposition 7.9. *Suppose that H is an amenable subgroup of G that contains a normal infinite cyclic subgroup. Let $G \overset{\alpha}{\curvearrowright} X$ be a weakly mixing minimal action on the Cantor set X with $M_\alpha(X) \neq \emptyset$. Then there is a free strictly ergodic action $H \overset{\gamma}{\curvearrowright} X$ such that if $G \overset{\beta}{\curvearrowright} X$ is any action extending γ then the diagonal action $G \overset{\alpha\times\beta}{\curvearrowright} X \times X$ is minimal and weakly mixing.*

Proof. Writing $H \overset{\alpha'}{\curvearrowright} X$ for the restriction of α to H, by Zorn's lemma there exists a nonempty closed set $A \subseteq X$ such that the restriction $\tilde{\alpha}$ of α' to A is minimal. By the amenability of H there exists a $\mu \in M_H(A)$, which we may take to be ergodic. This μ is either atomic (which, in view of minimality, occurs precisely when A is finite) or atomless.

Suppose first that μ is atomless. Write $\mathscr{C}$ for the countable collection of nonempty clopen subsets of X. Since α is minimal, each measure in $M_\alpha(X)$ has full support. Since α is weakly mixing, it follows by Lemma 7.7 that for every $T = (U_1, U_2, V_1, V_2) \in \mathscr{C}^4$ there exists an $s_T \in G$ such that the set H_T of all $h \in H$ for which $hs_T \in N_\alpha(U_1, U_2) \cap N_\alpha(V_1, V_2)$ is infinite. Note that $\mathscr{C}^4$ is countable and a countable intersection of dense G_δ sets in a Polish space is again a dense G_δ. With (Z, ζ) denoting as usual a fixed standard atomless probability space, Lemmas 7.6 and 7.8 imply that the set of free ergodic p.m.p. actions $\gamma \in \mathrm{Act}(H, Z, \zeta)$ that are disjoint from $H \overset{\tilde{\alpha}}{\curvearrowright} (A, \mu)$ and satisfy $N_{\gamma,\varepsilon}(W_1, W_2) \cap N_{\gamma,\varepsilon}(Z_1, Z_2) \cap H_T \neq \emptyset$ for all nonnull measurable sets $W_1, W_2, Z_1, Z_2 \subseteq A$, $\varepsilon > 0$, and $T = (U_1, U_2, V_1, V_2) \in \mathscr{C}^4$ contains a dense G_δ set (the weak mixing aspect of Lemma 7.8 is not needed here but will be used later in Section 10). In particular there exists such an action $H \overset{\gamma}{\curvearrowright} (Z, \zeta)$. By the Jewett–Krieger theorem [74, 83] we may assume that Z is equal to our Cantor set X and that γ is a free strictly ergodic action on X with ζ being the unique element in $M_\gamma(X)$ (note that the compact metrizable space produced by the construction in [83] is zero-dimensional and by the infiniteness of H cannot have isolated points, so that it must be the Cantor set). This implies by Lemma 7.2 that the actions $H \overset{\tilde{\alpha}}{\curvearrowright} A$ and $H \overset{\gamma}{\curvearrowright} X$ are disjoint. It then follows by Lemma 7.1 that if $G \overset{\beta}{\curvearrowright} X$ is any action extending γ then the diagonal action $G \overset{\alpha\times\beta}{\curvearrowright} X \times X$ is minimal.

Let us also check that such a diagonal action $\alpha \times \beta$ with β extending γ is weakly mixing. Let $U_1, U_2, V_1, V_2, W_1, W_2, Z_1, Z_2 \subseteq X$ be nonempty clopen sets. Write $T = (U_1, U_2, V_1, V_2)$. By our choice of γ we can find an $h \in H_T$, i.e., $hs_T \in N_\alpha(U_1, U_2) \cap N_\alpha(V_1, V_2)$, such that for all $\varepsilon > 0$ we have $h \in N_{\gamma,\varepsilon}(\beta_{s_T} W_1, W_2) \cap N_{\gamma,\varepsilon}(\beta_{s_T} Z_1, Z_2)$, that is, $\zeta(\beta_{hs_T} W_1 \cap W_2) > \zeta(\beta_{s_T} W_1)\zeta(W_2) - \varepsilon$ and $\zeta(\beta_{hs_T} Z_1 \cap Z_2) > \zeta(\beta_{s_T} Z_1)\zeta(Z_2) - \varepsilon$. Since the measure ζ has full support by the minimality of γ, it follows that $hs_T \in N_\beta(W_1, W_2) \cap N_\beta(Z_1, Z_2)$. Hence $hs_T \in N_{\alpha\times\beta}(U_1 \times W_1, U_2 \times W_2) \cap N_{\alpha\times\beta}(V_1 \times Z_1, V_2 \times Z_2)$. Since products of clopen sets form a basis for the topology on $X \times Y$, we conclude that $\alpha \times \beta$ is weakly mixing.

Suppose now that μ is atomic, in which case A is finite. We can no longer apply the above argument to $\tilde{\alpha}$ itself, but we can apply it instead to the diagonal action $H \overset{\tilde{\alpha}\times\mathrm{tr}}{\curvearrowright} (A \times X, \mu \times \rho)$, where tr denotes the trivial action and X is equipped with some atomless Borel probability measure ρ. This yields a free strictly ergodic action $H \overset{\gamma}{\curvearrowright} X$ which, with respect to its unique invariant Borel probability measure ζ, is disjoint from $H \overset{\tilde{\alpha}\times\mathrm{tr}}{\curvearrowright} (A \times X, \mu \times \rho)$ and hence also from $H \overset{\tilde{\alpha}}{\curvearrowright} (A, \mu)$, and, with H_T as above, satisfies $N_{\gamma,\varepsilon}(W_1, W_2) \cap N_{\gamma,\varepsilon}(Z_1, Z_2) \cap H_T \neq \emptyset$ for all nonnull measurable sets $W_1, W_2, Z_1, Z_2 \subseteq X$, $\varepsilon > 0$, and $T = (U_1, U_2, V_1, V_2) \in \mathscr{C}^4$. We can now conclude, arguing as before, that for every action $G \overset{\beta}{\curvearrowright} X$ extending γ the diagonal action $G \overset{\alpha\times\beta}{\curvearrowright} X \times X$ is minimal and weakly mixing. □

We remark that the action β extending γ that is universally quantified in Proposition 7.9 need not admit any invariant Borel probability measures, even though γ does admit one. In our applications in Proposition 8.5 and Lemma 9.1 we will be handling β that do satisfy $M_\beta(X) \neq \emptyset$, specifically ones that will be given to us through the use of Proposition 7.12. In this case $M_{\alpha\times\beta}(X \times X)$ will be nonempty as it will contain at least one product measure. What is interesting in the proof of Proposition 7.9, especially in the context of these applications, is the auxiliary nature of the use of p.m.p. ergodic theory via the the amenability of the subgroup H. It could very well happen that the α'-minimal set A on which we apply this ergodic theory is null for all α-invariant Borel probability measures.

If H is itself an infinite cyclic subgroup of G in Proposition 7.9, for example if we replace the original H with the hypothesized normal infinite cyclic subgroup, then the conclusion is valid with no extra assumptions on H. One may wonder whether this special form of the proposition is sufficient for all practical purposes, but the problem in our applications in Sections 8 and 9 is that we do not have a mechanism for extending an action of the normal infinite cyclic subgroup of H to all of G, or more specifically of first extending the action to H. So the general version will be necessary.

We now turn to the proof of Proposition 7.12, which as mentioned above provides the dynamical data that will be fed into Proposition 7.9 in subsequent sections. We will need the following asymptotic freeness result involving random permutation matrices due to Collins and Dykema [13]. It has the consequence that if we have actions of two groups G and H on a common finite set of large cardinality then we can randomly conjugate one of the actions to obtain, with high probability, an action of $G*H$ that is almost free on a given set of words whose individual letters act almost freely. This principle also works more generally for sofic approximations, in which the axioms for a group action are allowed to be corrupted on a proportionally small part of the set. In fact the original motivation in [13] was to establish the soficity of certain amalgamated free products. Our application in Lemma 7.11 below has a similar sofic spirit.

Theorem 7.10. [13, Theorem 2.1] *Write S_n for the set of $n \times n$ permutation matrices in M_n and* tr *for the normalized trace on M_n. Then for all $n, d \in \mathbb{N}$ and $A_1, \ldots, A_{2d} \in S_n$ one has*

$$\frac{1}{n!} \sum_{U \in S_n} \mathrm{tr}(A_1(UA_2U^*)A_3(UA_4U^*) \cdots A_{2d-1}(UA_{2d}U^*))$$

$$< r_d \max_{1 \le i \le 2d} \mathrm{tr}(A_i) + \frac{t_d}{n}.$$

where $r_d, t_d > 0$ are constants depending only on d.

Note that if we view an $n \times n$ permutation matrix A as a permutation of a set with n elements then its normalized trace is exactly the number of fixed points of this permutation divided by n.

Lemma 7.11. *Suppose that G is residually finite and that H is countably infinite and amenable. Let $\varepsilon > 0$ and let $F \subseteq G*H$ be a nonempty finite set. Define F_H to be the set of all elements in H (including $e \in H$) that appear as letters among the reduced words in $G*H$ that belong to $F^{-1}F$. Then there exist a (F_H, ε)-invariant finite set $K \subseteq H$, an action $G \overset{\rho}{\curvearrowright} K$, and permutations $\{\sigma_s\}_{s\in F_H}$ and τ of K such that if $w = h_1g_1 \cdots h_kg_k$ is a nontrivial reduced word in $G*H$ that belongs to $F^{-1}F$ then*

$$\mathrm{tr}(\sigma_{h_1}(\tau\rho_{g_1}\tau^{-1})\sigma_{h_2}(\tau\rho_{g_2}\tau^{-1}) \cdots \sigma_{h_k}(\tau\rho_{g_k}\tau^{-1})) < \varepsilon.$$

The same conclusion holds for words in $F^{-1}F$ that start and end with elements both in G, both in H, or in G and H, respectively.

Proof. We assume, without loss of generality, that $\varepsilon < 1/|F^{-1}F|$. Write F_G for the set of all elements of G (including $e \in G$) that appear as letters in reduced words in $G*H$ that belong to $F^{-1}F$. Since G is residually finite, there exists a finite-index normal subgroup N of G that does not contain any element of $F_G^2 \setminus \{e\}$. Let $\ell(w)$ denote the length of a reduced word w in $G*H$. For $d \in \mathbb{N}$ let $r_d, t_d \geq 1$ be the constants given by Theorem 7.10, and define $L = \max_{w\in F^{-1}F}\lceil \ell(w)/2\rceil$, $R = \max_{1\leq d\leq L} r_d$, and $T = \max_{1\leq d\leq L} t_d$. Since H is amenable and infinite, there exists a finite set $K \subseteq H$ such that

(i) K is $(F_H^2, \varepsilon^2/2R)$-invariant, and in particular (F_H, ε)-invariant,
(ii) $|K| > 2T/\varepsilon^2$,
(iii) $|K|! > |F^{-1}F|/(1 - |F^{-1}F|\varepsilon)$, and
(iv) $m := |K|/[G : N]$ is a positive integer.

Consider the diagonal action $G \curvearrowright G/N \times \{1, \dots, m\}$ coming from left translation on the first factor and the trivial action on the second. By (iv) there is a bijection $\theta : K \to G/N\times\{1, \dots, m\}$. Define an action $G \overset{\rho}{\curvearrowright} K$ by $\rho_s h = \theta^{-1}s\theta h$ for all $s \in G$ and $h \in K$. By our choice of N, we conclude that $\mathrm{tr}(\rho_s) = 0$ whenever $s \notin N$. For every $s \in F_H$ define a permutation σ_s of K by setting $\sigma_s h = sh$ for $h \in s^{-1}K \cap K$ and on $K \setminus s^{-1}K$ taking σ_s to be an arbitrary bijection with $K \setminus sK$ that agrees with $\sigma_{s^{-1}}^{-1}$ if $s^{-1} \in F_H$ and $\sigma_{s^{-1}}$ was already defined (this can be guaranteed if the maps σ_s are defined recursively, since F_H is a finite set). By (i), for all $s \in F_H \setminus \{e\}$ and $s_1, s_2 \in F_H$ such that $\sigma_{s_1}\sigma_{s_2} \neq \mathrm{id}_K$ we have $\mathrm{tr}(\sigma_s) \leq |K \setminus s^{-1}K|/|K| < \varepsilon^2/2R$ and

$$\mathrm{tr}(\sigma_{s_1}\sigma_{s_2}) \leq \frac{|K \setminus \bigcap_{s\in(F_H)^2} s^{-1}K|}{|K|} < \frac{\varepsilon^2}{2R}. \tag{7.4}$$

Let $w = h_1g_1 \cdots h_kg_k \in F^{-1}F$ be a nontrivial word in its reduced form. Identifying the permutations σ_s with elements in the set S_K of permutation matrices in the matrix algebra M_K with entries indexed by pairs in K, we apply Theorem 7.10 to get

$$\frac{1}{|K|!}\sum_{U\in S_K} \mathrm{tr}(\sigma_{h_1}(U\rho_{g_1}U^*)\sigma_{h_2}(U\rho_{g_2}U^*) \cdots \sigma_{h_k}(U\rho_{g_k}U^*)) < r_k\frac{\varepsilon^2}{2R} + \frac{t_k}{|K|} \overset{\text{(ii)}}{<} \varepsilon^2.$$

Since we are averaging positive real numbers, an easy combinatorial argument shows that for at least $\lfloor |K|!(1 - \varepsilon)\rfloor$ elements $U \in S_K$ one has that

$$\mathrm{tr}(\sigma_{h_1}(U\rho_{g_1}U^*)\sigma_{h_2}(U\rho_{g_2}U^*) \cdots \sigma_{h_k}(U\rho_{g_k}U^*)) < \varepsilon.$$

If $w = g_1h_1 \cdots g_kh_k \in F^{-1}F$ is a nontrivial word in its reduced form, then analogously we can find at least $\lfloor |K|!(1-\varepsilon)\rfloor$ elements $U \in S_K$ for which

$$\mathrm{tr}(\sigma_{h_k}(U\rho_{g_1}U^*)\sigma_{h_1}(U\rho_{g_2}U^*)\cdots\sigma_{h_{k-1}}(U\rho_{g_k}U^*)) < \varepsilon,$$

and hence also

$$\mathrm{tr}((U\rho_{g_1}U^*)\sigma_{h_1}(U\rho_{g_2}U^*)\cdots\sigma_{h_{k-1}}(U\rho_{g_k}U^*)\sigma_{h_k}) < \varepsilon.$$

If $w \in F^{-1}F\backslash\{e\}$ has reduced form $w = g_1h_1 \cdots g_{k-1}h_{k-1}g_k$ and $g_kg_1 \neq e$, then $g_kg_1 \in (F_G)^2\backslash\{e\}$ and so does not belong to N. Hence

$$\begin{aligned}\frac{1}{|K|!}&\sum_{U\in S_K}\mathrm{tr}((U\rho_{g_1}U^*)\sigma_{h_1}(U\rho_{g_2}U^*)\cdots(U\rho_{g_{k-1}}U^*)\sigma_{h_{k-1}}(U\rho_{g_k}U^*))\\ &= \frac{1}{|K|!}\sum_{U\in S_K}\mathrm{tr}((U\rho_{g_kg_1}U^*)\sigma_{h_1}(U\rho_{g_2}U^*)\cdots(U\rho_{g_{k-1}}U^*)\sigma_{h_{k-1}})\\ &< r_{k-1}\frac{\varepsilon^2}{2R} + \frac{t_{k-1}}{|K|} \overset{\text{(ii)}}{<} \varepsilon^2.\end{aligned}$$

We conclude again that for at least $\lfloor |K|!(1-\varepsilon)\rfloor$ elements $U \in S_K$ one has that

$$\mathrm{tr}((U\rho_{g_1}U^*)\sigma_{h_1}(U\rho_{g_2}U^*)\cdots\sigma_{h_{k-1}}(U\rho_{g_k}U^*)) < \varepsilon.$$

If $g_kg_1 = e$, then we need to bound an expression of the form

$$\frac{1}{|K|!}\sum_{U\in S_K}\mathrm{tr}(\sigma_{h_1}(U\rho_{g_2}U^*)\cdots(U\rho_{g_{k-1}}U^*)\sigma_{h_{k-1}}).$$

It will become clear how to do this once we treat the case of nontrivial words in $F^{-1}F$ that have reduced form that begins and ends with elements of H, i.e., $w \in F^{-1}F$ of the form $w = h_1g_1 \cdots h_{k-1}g_{k-1}h_k$. In this scenario, if it happens that $\sigma_{h_k}\sigma_{h_1} \neq \mathrm{id}_K$ then we treat $\sigma_{h_k}\sigma_{h_1}$ as a single permutation and apply (7.4) and Theorem 7.10 to obtain

$$\begin{aligned}\frac{1}{|K|!}&\sum_{U\in S_K}\mathrm{tr}(\sigma_{h_1}(U\rho_{g_1}U^*)\cdots\sigma_{h_{k-1}}(U\rho_{g_{k-1}}U^*)\sigma_{h_k})\\ &= \frac{1}{|K|!}\sum_{U\in S_K}\mathrm{tr}(\sigma_{h_k}\sigma_{h_1}(U\rho_{g_1}U^*)\cdots\sigma_{h_{k-1}}(U\rho_{g_{k-1}}U^*))\\ &< r_{k-1}\frac{\varepsilon^2}{2R} + \frac{t_{k-1}}{|K|} < \varepsilon^2.\end{aligned}$$

If on the other hand $\sigma_{h_k}\sigma_{h_1} = \mathrm{id}_K$, then we have fewer (i.e., $2k-3$) permutations at play in the application of Theorem 7.10 to a word beginning and ending with elements of G. We can continue in this way recursively, with a smaller amount of permutations at each step until either "cancellation" of the above nature does not occur, or we end up with a single permutation (some ρ_g with $g \in F_G$ or some σ_h with $h \in F_H$), in which case the inequality is clearly satisfied.

Now for each $w \in F^{-1}F \setminus \{e\}$ we found at least $\lfloor |K|!(1-\varepsilon)\rfloor$ elements $U \in S_K$ that satisfy a certain desired inequality. We claim that there exists $\tau \in S_K$ that achieves this simultaneously for all $w \in F^{-1}F \setminus \{e\}$. This is a consequence of the following counting argument: for each nontrivial $w \in F^{-1}F$ there are at most $|K|! - \lfloor |K|!(1-\varepsilon)\rfloor$ elements in S_K that do not satisfy the desired inequality associated with w. However, from (iii) it follows that $|F^{-1}F| \cdot (|K|! -$

$\lfloor |K|!(1-\varepsilon)\rfloor) < |K|!$. As there are $|K|!$ permutations in S_K, we obtain the existence of a $\tau \in S_K$ satisfying the condition in the corollary statement. □

Since an action of an infinite amenable group on the Cantor set has the URP if and only if it is essentially free (see [26, Theorem 3.6]), the following is a strengthening of Theorem 5.7 in the case of the Cantor set.

Proposition 7.12. *Suppose that X is the Cantor set, G is residually finite, H is a countably infinite amenable group, and $H \curvearrowright X$ an essentially free minimal action that has comparison. Let F be a finite subset of $G*H$. Then there is an action $G*H \curvearrowright X$ extending $H \curvearrowright X$ (viewing H as a subgroup of $G*H$) such that*

(i) the action is (O_1, O_2, E)-squarely divisible for all nonempty clopen sets $O_1, O_2 \subseteq X$ and finite sets $E \subseteq G*H$ containing e,
(ii) $M_{G*H}(X) = M_H(X)$, and
(iii) there exists a nonempty clopen set $A \subseteq X$ such that (F, A) is a tower.

Proof. Let $e \in F_G \subseteq G$ and $e \in F_H \subseteq H$ be the finite sets as defined in the statement of Lemma 7.11 and the beginning of its proof. Applying Lemma 7.11, we find a (F_H, ε)-invariant finite set $K \subseteq H$, an action $G \overset{\rho}{\curvearrowright} K$, and permutations $\{\sigma_s\}_{s\in F_H}$ and τ of K such that if $w = h_1 g_1 \cdots h_k g_k$ is a nontrivial reduced word in $F^{-1}F$ then the permutation

$$\rho_w := \sigma_{h_1}(\tau\rho_{g_1}\tau^{-1})\sigma_{h_2}(\tau\rho_{g_2}\tau^{-1})\cdots\sigma_{h_k}(\tau\rho_{g_k}\tau^{-1}) \tag{7.5}$$

satisfies $\mathrm{tr}(\rho_w) < \varepsilon$ for some $\varepsilon < 1/(|F^{-1}F| + L)$ where $L = \max_{w\in F^{-1}F} \ell(w)$ and $\ell(w)$ denotes the length of the reduced word $w \in F^{-1}F$. The same conclusion holds for words in $F^{-1}F$ that start and end with elements both in G, both in H, or in G and H, respectively. We also recall from the proof of Lemma 7.11 that $\sigma_s h = sh$ whenever $s \in F_H$ and $h \in s^{-1}K \cap K$, and that for some finite-index normal subgroup N of G one has $\rho_g = \mathrm{id}_K$ for every $g \in N$.

For $w = h_1 g_1 \cdots h_k g_k \in F^{-1}F \setminus \{e\}$ and $1 \le p \le k$ define the permutation $\varphi_{w,p} = (\tau\rho_{g_p}\tau^{-1})\sigma_{h_{p+1}}(\tau\rho_{g_{p+1}}\tau^{-1})\cdots\sigma_{h_k}(\tau\rho_{g_k}\tau^{-1})$ of K. It is straightforward to check that if $h \in \bigcap_{p=1}^{k} \varphi_{w,p}^{-1}(K^{F_H})$, then

$$\rho_w h = h_1(\tau\rho_{g_1}\tau^{-1})h_2(\tau\rho_{g_2}\tau^{-1})\cdots h_k(\tau\rho_{g_k}\tau^{-1})h,$$

where by $h_1, \dots, h_k$ we mean the action of these elements by left translation. One may define similarly $\varphi_{w,p}$ for any $w \in F^{-1}F \setminus \{e\}$, with the range of p depending on the length of w but uniformly bounded by L. Using $\mathrm{Fix}(\cdot)$ to denote the fixed-point set, consider the intersection

$$W := \Bigg(\bigcap_{w\in F^{-1}F}\bigcap_{p}\varphi_{w,p}^{-1}(K^{F_H})\Bigg) \cap \Bigg(\bigcap_{w\in F^{-1}F\setminus\{e\}} \mathrm{Fix}(\rho_w)^c\Bigg).$$

Observe that, for every $h \in W$, when we apply the permutations defining ρ_w in sequence, each of the permutations σ_{h_i} acts as translation by h_i. Moreover, since the $\varphi_{w,p}$ are permutations and K is (F_H, ε)-invariant, the complement of W consists of at most $|L||F^{-1}F|\varepsilon|K| + |F^{-1}F|\varepsilon|K|$ elements, which, by our choice of ε, is less than $|K|$. Hence W is nonempty and we may choose an $h_0 \in W$.

By the topological freeness of the action $H \curvearrowright X$ we can find a nonempty clopen set $A \subseteq X$ such that (K, A) is a tower. Define an action $G*H \curvearrowright X$ extending the H-action by declaring it on an element $g \in G$ to be given by

$$ghx = \tau\rho_g\tau^{-1}hx$$

for all $x \in A$ and $h \in K$ and $gx = x$ for all $x \in X \setminus KA$. Thus the elements of G act on KA by permuting the levels of the tower (K, A) in a way that preserves the K-orbits of points in A, i.e., $gKx = Kx$ for all $x \in A$. This has the effect $M_{G*H}(X) = M_H(X)$. Indeed if we are given a $\mu \in M_H(X)$, a $g \in G$, and a Borel set $B \subseteq X$ then for every $k \in K$ we have $g(B \cap kA) = \tau\rho_g\tau^{-1}(k)(k^{-1}B \cap A)$ and hence, using the H-invariance of μ,

$$\mu(g(B \cap kA)) = \mu(\tau\rho_g\tau^{-1}(k)(k^{-1}B \cap A)) = \mu(k^{-1}B \cap A) = \mu(B \cap kA),$$

so that writing $B = (B \setminus KA) \sqcup (\bigsqcup_{k\in K} B \cap kA)$ and noting that $g(B \setminus KA) = B \setminus KA$ we obtain $\mu(gB) = \mu(B)$. It is moreover straightforward to verify that our choice of h_0 means that for every nontrivial element $w \in G * H$ belonging to $F^{-1}F$ and every $x \in A$ we have

$$w(h_0x) = \rho_w(h_0)x.$$

This, together with the fact that $h_0 \in \bigcap_{w\in F^{-1}F\setminus\{e\}} \mathrm{Fix}(\rho_w)^c$, implies that (F, h_0A) forms a tower. Thus conditions (ii) and (iii) of the lemma statement are fulfilled.

It remains to verify that the action $G * H \curvearrowright X$ satisfies condition (i). For this purpose we may regard it as an action of $D * H$ where D is the finite group G/N (since $\rho_g = \mathrm{id}_K$ for $g \in N$). Let E be a finite subset of $D * H$ containing e. Then there are a finite set $L \subseteq H$ containing e and an $n \in \mathbb{N}$ such that $E \subseteq (D \cup L)^n$. Set $W = (KK^{-1} \cup L)^n$.

Let $x \in X$. If $x \in h_1A$ for some $h_1 \in K$ then for every $s \in D$ there exists an $h_2 \in K$ (namely $\tau\rho_s\tau^{-1}(h_1)$) such that $sx = h_2h_1^{-1}x$, and if $x \notin KA$ then $sx = x$ for every $s \in D$. Therefore $Dx \subseteq KK^{-1}x$ and hence $Ex \subseteq Wx$. This means that for any set $Y \subseteq X$ one has

$$EY \subseteq WY \quad \text{and} \quad Y^W \subseteq Y^E. \tag{7.6}$$

By Theorem 5.7, the action $H \curvearrowright X$ is (O_1, O_2, W)-squarely divisible. Let $\{V_{i,j}\}_{i,j=1}^n$ and U be the sets witnessing this square divisibility (with associated boundary set B as in Definition 3.3). Then the sets $V_{i,j}$ will automatically be E-disjoint by equation (7.6). Again by equation (7.6), the associated boundary set $B = \overline{V} \cap ((V \cap \overline{U}^c)^W)^c$ contains the set $\overline{V} \cap ((V \cap \overline{U}^c)^E)^c$. It clearly follows that the sets $\{V_{i,j}\}_{i,j=1}^n$ also witness (O_1, O_2, E)-squarely divisibility for $D * H \curvearrowright X$. □

8. Spaces of actions

We introduce here certain subspaces of $\mathrm{Act}(G, X)$ that will play a central role in subsequent sections. Recall that $\mathrm{Act}(G, X)$ denotes the space of all actions $G \curvearrowright X$ with the topology of elementwise compact-open convergence.

In the case that X is the Cantor set, which will be the main focus of the rest of the paper, we make the following series of observations about $\mathrm{Act}(G, X)$ that will help us establish Proposition 8.4. Write $\mathscr{C}$ for the countable collection of nonempty clopen subsets of X.

First we remark that set of minimal actions in $\mathrm{Act}(G, X)$ is a G_δ. Indeed for every $A \in \mathscr{C}$ consider the set $\mathscr{W}_A$ of all $\alpha \in \mathrm{Act}(G, X)$ for which there exists a finite set $F \subseteq G$ such that $\bigcup_{s\in F} \alpha_sA = X$. Then $\mathscr{W}_A$ is open and the set of minimal actions is equal to $\bigcap_{A\in\mathscr{C}} \mathscr{W}_A$, which is a G_δ.

As observed in [14, Section 4], the set of free actions in $\mathrm{Act}(G, X)$ is a G_δ. The same is true for the set of topologically free actions: fixing an enumeration $s_1, s_2, \dots$ of $G \setminus \{e\}$, we observe that for every $n \in \mathbb{N}$ and $A \in \mathscr{C}$ the set $\mathscr{W}_{n,A}$ of all actions $\alpha \in \mathrm{Act}(G,X)$ such that there exists an $x \in A$ for which $\alpha_{s_n} x \neq x$ is open, and taking the intersection $\bigcap_{n\in\mathbb{N}} \bigcap_{A\in\mathscr{C}} \mathscr{W}_{n,A}$ we obtain precisely the set of all topologically free actions.

To see that the set of weak mixing actions in $\mathrm{Act}(G,X)$ is also a G_δ, for every $A_1, A_2, B_1, B_2 \in \mathscr{C}$ write $\mathscr{W}_{A_1,A_2,B_1,B_2}$ for the set of all $\alpha \in \mathrm{Act}(G,X)$ for which there exists $s \in G$ such that $\alpha_s A_1 \cap A_2 \neq \emptyset$ and $\alpha_s B_1 \cap B_2 \neq \emptyset$. Then $\mathscr{W}_{A_1,A_2,B_1,B_2}$ is open, and the set of all weak mixing actions is equal to $\bigcap_{A_1,A_2,B_1,B_2\in\mathscr{C}} \mathscr{W}_{A_1,A_2,B_1,B_2}$, which is a G_δ.

Finally we note that the set of actions $\alpha \in \mathrm{Act}(G,X)$ for which $M_\alpha(X) \neq \emptyset$ is closed in $\mathrm{Act}(G,X)$, and hence also a G_δ. To verify this, suppose that $(\alpha_j)_{j\in\mathbb{N}}$ is a sequence of elements in $\mathrm{Act}(G,X)$ which each admit an invariant Borel probability measure and converge to some $\alpha \in \mathrm{Act}(G,X)$. We must show that $M_\alpha(X) \neq \emptyset$. For each j pick a $\mu_j \in M_{\alpha_j}(X)$ and take a weak* cluster point $\mu \in M(X)$ of the sequence $(\mu_j)_{j\in\mathbb{N}}$, which exists by weak* compactness. We may assume, by passing to a subsequence if necessary, that μ_j converges to μ. We will show that $\mu \in M_\alpha(X)$. Let $A \in \mathscr{C}$, where $\mathscr{C}$ is as above, and let $s \in G$. Since the Borel σ-algebra of X is generated by $\mathscr{C}$, it suffices to show that $\mu(\alpha_s A) = \mu(A)$. Since α_j eventually belongs to the basic open set $U_{\alpha,\{s\},\mathscr{P}}$, where $\mathscr{P}$ is any clopen partition containing A, we may assume that $\alpha_{j,s} A = \alpha_s A$ for all $j \in \mathbb{N}$. Given that $1_A \in C(X)$ we now have

$$\lim_j \mu_j(A) = \lim_j \int_X 1_A d\mu_j = \int_X 1_A d\mu = \mu(A)$$

while

$$\begin{aligned}\lim_j \mu_j(A) = \lim_j \mu_j(\alpha_{j,s} A) &= \lim_j \mu_j(\alpha_s A) \\ &= \lim_j \int_X 1_{\alpha_s A} d\mu_j = \int_X 1_{\alpha_s A} d\mu = \mu(\alpha_s A).\end{aligned}$$

This implies $\mu(\alpha_s A) = \mu(A)$, as desired.

Definition 8.1. A transformation $T : X \to X$ is *spectrally aperiodic* if there is no clopen subset $A \subseteq X$ and $n \in \mathbb{N}$ such that the sets $A, TA, \dots, T^n A$ partition X.

Definition 8.2. Write $\mathrm{WA}(G,X)$ for the set of all minimal topologically free actions in $\mathrm{Act}(G,X)$ that are weakly mixing and satisfy $M_G(X) \neq \emptyset$. For $d \in \mathbb{N}$ write $\mathrm{A}^*(F_d,X)$ for the set of all topologically free actions in $\mathrm{Act}(F_d,X)$ with $M_{F_d}(X) \neq \emptyset$ that are strictly ergodic and spectrally aperiodic on each standard generator.

Remark 8.3. Note that every $\alpha \in \mathrm{A}^*(F_d,X)$ is itself strictly ergodic.

Proposition 8.4. *Suppose that G is infinite and X is the Cantor set. Then* $\mathrm{WA}(G,X)$ *and* $\mathrm{A}^*(F_d,X)$ *for $d \in \mathbb{N}$ are a nonempty G_δ subsets of the respective spaces* $\mathrm{Act}(G,X)$ *and* $\mathrm{Act}(F_d,X)$*. In particular, they are Polish spaces.*

Proof. The fact that $\mathrm{WA}(G,X)$ is a G_δ follows from the discussion before Definition 8.1.

By the discussion at the beginning of the section, the minimal transformations form a G_δ set in $\mathrm{Act}(\mathbb{Z},X)$. So do the uniquely ergodic actions in $\mathrm{Act}(\mathbb{Z},X)$, as shown in [40] (see Theorem 1.3 and the last part of the proof of Proposition 6.7 therein). Thus the set of strictly ergodic actions in $\mathrm{Act}(\mathbb{Z},X)$ is a G_δ. Moreover, the set of transformations $T \in \mathrm{Act}(\mathbb{Z},X)$ for which there exist a clopen set $A \subseteq X$ and an $n \in \mathbb{N}$ so that the sets $A, TA, \dots, T^n A$ partition X is clearly open, and the set of spectrally aperiodic actions is its complement and hence is a G_δ. Since the restriction map $\mathrm{Act}(F_d, X) \to \mathrm{Act}(\langle a\rangle, X) = \mathrm{Act}(\mathbb{Z}, X)$ is continuous for each of the standard generators a of F_d, we can intersect the inverse images of the above G_δ sets under these maps and then further intersect this with the set of topologically free actions in $\mathrm{Act}(F_d, X)$ satisfying $M_{F_d}(X) \neq \emptyset$ (which is a G_δ by the discussion before Definition 8.1) so as to express $\mathrm{A}^*(F_d, X)$ as the intersection of finitely many G_δ sets, which is again a G_δ set.

As explained in Example 3.17, every countably infinite group admits a free minimal mixing action on the Cantor set with $M_G(X) \neq \emptyset$, and so $\mathrm{WA}(G, X)$ is nonempty. The examples in Section 12 show that $\mathrm{A}^*(F_d, X)$ is nonempty for $d \in \mathbb{N}$ (the construction in Section 12 is done in the case $d = 2$ but can be carried out similarly for other d). □

Proposition 8.5. *Suppose X is the Cantor set. Suppose that G is residually finite and that H is amenable and contains a normal infinite cyclic subgroup. Then $\mathrm{WA}(G * H, X)$ is a dense G_δ set inside the space $\mathscr{M}$ of all weakly mixing minimal actions with $M_{G*H}(X) \neq \emptyset$.*

Proof. The space $\mathscr{M}$ is a G_δ subset of $\mathrm{Act}(G * H, X)$ by the discussion preceding Definition 8.1, and by definition $\mathrm{WA}(G * H, X)$ is the set of all actions in $\mathscr{M}$ that satisfy the additional requirement of topological freeness.

As we did at the beginning of the section, fix an enumeration $s_1, s_2, \dots$ of $G * H \setminus \{e\}$ and for every $n \in \mathbb{N}$ and A in the countable collection $\mathscr{C}$ of nonempty clopen subsets of X write $\mathscr{W}_{n,A}$ for the set of all actions $\alpha \in \mathscr{M}$ such that there exists an $x \in A$ for which $\alpha_{s_n} x \neq x$. Then $\mathscr{W}_{n,A}$ is an open set in $\mathscr{M}$ and $\mathrm{WA}(G * H, X) = \bigcap_{n\in\mathbb{N}} \bigcap_{A\in\mathscr{C}} \mathscr{W}_{n,A}$. We will show that each $\mathscr{W}_{n,A}$ is dense in $\mathscr{M}$, and the density of $\mathrm{WA}(G * H, X)$ in $\mathscr{M}$ will then be a consequence of the Baire category theorem.

To verify the density of $\mathscr{W}_{n,A}$ in $\mathscr{M}$, let $\alpha \in \mathscr{M}$. Take a free minimal action $H \curvearrowright X$ as given by Proposition 7.9. This action has comparison by [58, Theorem A] together with [50, Theorem 6.1]. By Proposition 7.12 (ignoring the first condition in its statement concerning square divisibility) there is an action $G * H \overset{\beta}{\curvearrowright} X$ extending $H \curvearrowright X$ such that $M_\beta(X)$ is equal to $M_H(X)$ (which is nonempty by the amenability of H) and such that there exists a $y \in X$ with $\beta_{s_n} y \neq y$.

The diagonal action $G * H \overset{\alpha\times\beta}{\curvearrowright} X \times X$ is minimal and weakly mixing by our choice of $H \curvearrowright X$ via Proposition 7.9. By assumption $M_\alpha(X) \neq \emptyset$, and so $M_{\alpha\times\beta}(X \times X)$ contains at least one product measure and in particular is nonempty.

By Proposition 2.1 (applied with respect to the canonical projection $h : X \times X \to X$ onto the first coordinate) we can conjugate $\alpha \times \beta$ to an action $\rho = g \circ (\alpha \times \beta) \circ g^{-1}$ on X which is as close as we wish to α in such a way that the image of $A \times X$ under the conjugacy g is equal to A. Since any point $x \in g(A \times \{y\}) \subseteq A$ has the property that $\rho_{s_n} x \neq x$, we have $\rho \in \mathscr{W}_{n,A}$, establishing density. □

In contrast to actions in $\mathrm{A}^*(F_d, X)$, which are minimal on generators by definition, an action in $\mathrm{WA}(F_d, X)$, despite being minimal, is often very far from being minimal on generators, as demonstrated by the following proposition.

Proposition 8.6. *Suppose X is the Cantor set. Let $d \geq 2$. The set $\mathscr{V}$ of all actions $F_d = \langle a_1, \dots, a_d\rangle \curvearrowright X$ in $\mathrm{WA}(F_d, X)$ such that each of the restriction actions $\langle a_k\rangle \curvearrowright X$ for $k = 1, \dots, d$ factors onto the trivial action on the Cantor set is comeagre.*

Proof. As usual denote by $\mathscr{C}$ the countable collection of nonempty clopen subsets of X. For each $k = 1, \dots, d$ and $A \in \mathscr{C}$ write $\mathscr{W}_{k,A}$ for the set of all actions α in $\mathrm{WA}(F_d, X)$ such that if $\alpha_{a_k} A = A$ then A can be partitioned into two clopen subsets that are fixed by α_{a_k}, i.e., there exist disjoint sets $A_1, A_2 \in \mathscr{C}$ such that $A = A_1 \sqcup A_2$ and $\alpha_{a_k} A_i = A_i$, for $i = 1, 2$ (if $\alpha_{a_k} A \neq A$ then α automatically belongs to $\mathscr{W}_{k,A}$). The set $\mathscr{W}_{k,A}$ is clearly open by the definition of the topology on $\mathrm{WA}(F_d, X)$. The set $\mathscr{W} := \bigcap_{k=1}^{d} \bigcap_{A \in \mathscr{C}} \mathscr{W}_{k,A}$ is a G_δ. We show that $\mathscr{W} \subseteq \mathscr{V}$. Let $\alpha \in \mathscr{W}$. Given $1 \leq k \leq d$, we set $\mathscr{P}_{k,0} = \{X\}$ and for $n \in \mathbb{N}$ recursively find a partition $\mathscr{P}_{k,n}$ of cardinality 2^n refining $\mathscr{P}_{k,n-1}$ by partitioning every set in $\mathscr{P}_{k,n-1}$ into two disjoint nonempty clopen sets each of which is fixed by α_{a_k}. Take the C*-subalgebra $A_k \subseteq C(X)$ generated by the indicator functions of the members of the partitions $\mathscr{P}_{k,n}$ over all $n \in \mathbb{N}$. Set $A_{k,n} = C^*(\{1_U : U \in \mathscr{P}_{k,n}\})$. Then $A_k = \varinjlim_n A_{k,n}$, with the canonical inclusions as connecting maps. It is then straightforward to check that $A_{k,n} \cong \mathbb{C}^{2^n}$, and, under a suitable isomorphism, the inclusion maps correspond to the injective maps $\varphi_{n+1,n} : \mathbb{C}^{2^n} \to \mathbb{C}^{2^{n+1}}$ given by $z \mapsto (z, z)$. Thus $A_k \cong \varinjlim_n (\mathbb{C}^{2^n}, \varphi_{n+1,n}) \cong C(X)$, see [16, Example III.2.5]. From this we obtain an equivariant factor map $(X, \alpha_{a_k}) \to (X, \mathrm{tr})$. It follows that $\mathscr{W} \subseteq \mathscr{V}$.

It remains to show that for every $A \in \mathscr{C}$, the set $\mathscr{W}_A = \bigcap_{k=1}^{d} \mathscr{W}_{k,A}$ is dense in $\mathrm{WA}(F_d, X)$. By the Baire category theorem it would then follow that $\mathscr{W}$ is a dense G_δ set contained in $\mathscr{V}$, so that $\mathscr{V}$ is comeagre. Let $A \in \mathscr{C}$ and let $\alpha \in \mathrm{WA}(F_d, X)$. We will first construct an action $\alpha^{(1)} \in \bigcap_{k=2}^{d} \mathscr{W}_{k,A}$ that is as close to α as we wish. As $\langle a_1 \rangle \cong \mathbb{Z}$ is amenable, by Proposition 7.9 there is an action $\langle a_1 \rangle \overset{\gamma}{\curvearrowright} X$ such that, letting $F_d \overset{\beta}{\curvearrowright} X$ be the action determined by $\beta_{a_1} x = \gamma_{a_1} x$ for all $x \in X$ and $\beta_{a_k} x = x$ for all $x \in X$ and $k = 2, \dots, d$, the diagonal action $\alpha \times \beta$ of F_d on $X \times X$ is minimal and weakly mixing. Since α is topologically free, so is $\alpha \times \beta$. Since every Borel probability measure on X that is γ-invariant is also β-invariant, $M_{\alpha \times \beta}(X \times X)$ contains at least one product measure and hence is nonempty. Thus $\alpha \times \beta$ belongs to $\mathrm{WA}(F_d, X \times X)$.

Let $g : X \times X \to X$ be any homeomorphism which satisfies $g(A \times X) = A$. We claim that the conjugated action $g \circ (\alpha \times \beta) \circ g^{-1}$ on X lies in $\bigcap_{k=2}^{d} \mathscr{W}_{k,A}$. Indeed, for $2 \leq k \leq d$, assume that $g \circ (\alpha \times \beta)_{a_k} \circ g^{-1}(A) = A$. We must show that A partition nontrivially into two disjoint clopen sets each of which is fixed by $g \circ (\alpha \times \beta)_{a_k} \circ g^{-1}$. Taking C to be any proper nonempty clopen subset of X, it is easy to see that $A = g(A \times C) \sqcup g(A \times (X \setminus C))$ is partition of the desired type.

By Proposition 2.1 (applied with respect to the canonical projection $X \times X \to X$ onto the first coordinate) we can find a conjugate $\alpha^{(1)}$ of $\alpha \times \beta$ as close as we wish to α, via a conjugation map which satisfies the properties as stated for g above.

Now repeat the above construction substituting a_2 for a_1 and $\alpha^{(1)}$ for α. Then we obtain an action $\alpha^{(2)} \in \mathrm{WA}(F_d, X)$ that in particular lies in $\mathscr{W}_{1,A}$ (even lies in $\mathscr{W}_{k,A}$ for all $k \neq 2$) and is as close as we wish to $\alpha^{(1)}$. Note that in our construction $\alpha^{(2)}$ was achieved as a conjugate (via a homeomorphism which maps $A \times X$ to A) of a product of $\alpha^{(1)}$ with another action. Since $\alpha^{(1)} \in \bigcap_{k=2}^{d} \mathscr{W}_{k,A}$, it follows by the nature of the product construction that $\alpha^{(2)}$ must also belong to $\bigcap_{k=2}^{d} \mathscr{W}_{k,A}$. Therefore $\alpha^{(2)}$ belongs to $\mathscr{W}_A = \bigcap_{k=1}^{d} \mathscr{W}_{k,A}$, establishing the desired density. □

9. Generic square divisibility I

The proof of our first genericity result, Theorem 9.2, rests largely on the density statement for O-square divisibility captured in Lemma 9.1. The main technical tools have already been developed in Section 7 in the form of Propositions 7.9 and 7.12, which permit us to construct a diagonal action that inherits the properties of minimality, weak mixing, and the existence of an invariant Borel probability measure from one of the factors while also possessing the desired degree of square divisibility. With such diagonal actions at hand we can then appeal to Proposition 2.1 to clinch the desired density. Without the requirements of weak mixing and the existence of an invariant Borel probability measure the construction would be much simpler, as we could pass to a minimal subsystem after taking a product, an operation that preserves neither of the two properties in general.

Lemma 9.1. *Suppose that G is residually finite and that H is infinite and amenable and contains a normal infinite cyclic subgroup. Suppose X is the Cantor set, and let O be a nonempty clopen subset of X. Then the set of O-squarely divisible actions in* $\mathrm{WA}(G*H, X)$ *is dense.*

Proof. Let $\alpha \in \mathrm{WA}(G*H, X)$ and let us show that we can approximate α arbitrarily well by an O-squarely divisible action in $\mathrm{WA}(G*H, X)$. By minimality, continuity, and the fact that X has no isolated points, we can find for every $x \in X$ a clopen neighbourhood A_x of x in X and an $s_x \in G*H \setminus \{e\}$ such that $\alpha_{s_x} A_x \subseteq O$. Then $\{A_x\}_{x\in X}$ is a clopen cover of X and hence by compactness has a finite subcover. Disjointifying this cover by a standard recursive process and taking unions of members of the resulting clopen partition for which the corresponding group element s_x is the same, we obtain a finite set $F \subseteq G*H \setminus \{e\}$ and a clopen partition $\{W_s\}_{s\in F}$ of X such that $\alpha_s W_s \subseteq O$ for every $s \in F$.

Take an action $H \curvearrowright X$ as given by Proposition 7.9. This action has comparison by [58, Theorem A] together with [50, Theorem 6.1]. By Proposition 7.12 there is an action $G*H \overset{\beta}{\curvearrowright} X$ extending $H \curvearrowright X$ (viewing H as a subgroup of $G*H$) such that β is (O_1, O_2, E)-squarely divisible for all nonempty clopen sets $O_1, O_2 \subseteq X$ and finite sets $E \subseteq G*H$ containing e, $M_{G*H}(X)$ is equal to $M_H(X)$ (which is nonempty by the amenability of H), and there exists a nonempty clopen set $A \subseteq X$ such that $(F \sqcup \{e\}, A)$ is a tower.

Form the diagonal action $G*H \overset{\alpha\times\beta}{\curvearrowright} X \times X$, which is minimal and weakly mixing by our choice of $H \curvearrowright X$ via Proposition 7.9.

Fix two disjoint nonempty clopen subsets O_1 and O_2 of A (which exist since X has no isolated points). Let $E \subseteq G*H$ be a finite subset containing e. As β is (O_1, O_2, E)-squarely divisible, by Proposition 3.9 there exist $n \in \mathbb{N}$, a collection $\{V_{i,j}\}_{i,j=1}^n$ of pairwise equivalent and pairwise E-disjoint clopen subsets of X, and, writing $V = \bigsqcup_{i,j=1}^n V_{i,j}$, $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = V^c$, and $B = V \cap (V^E)^c$, one has the following:

(i) $V_{i,1} \prec_\beta O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ for every $i = 1, \dots, n$,
(ii) $R \prec_\beta O_2 \cap V \cap (V_1 \cup B)^c$,
(iii) $B \prec_\beta O_2 \cap R$.

Denoting by C' the product set $X \times C$ for every $C \subseteq X$, we obtain from this

(i) $V'_{i,1} \prec_{\alpha\times\beta} O'_1 \cap \bigsqcup_{j=2}^n V'_{i,j} \cap (B')^c$ for every $i = 1, \dots, n$,
(ii) $R' \prec_{\alpha\times\beta} O'_2 \cap V' \cap (V'_1 \cup B')^c$,
(iii) $B' \prec_{\alpha\times\beta} O'_2 \cap R'$.

Now since the sets $\beta_s(O_1 \sqcup O_2)$ for $s \in F$ are pairwise disjoint subsets of $X \setminus (O_1 \sqcup O_2)$ (since $e \notin F$) and $\alpha_s W_s \subseteq O$ for every $s \in F$, we have

$$X \times (O_1 \sqcup O_2) = \bigsqcup_{s\in F} W_s \times (O_1 \sqcup O_2)$$

$$\prec_{\alpha\times\beta} \bigsqcup_{s\in F} (\alpha\times\beta)_s(W_s \times (O_1 \sqcup O_2))$$
$$\subseteq O \times (X \setminus (O_1 \sqcup O_2)).$$

This shows that $O_1' \sqcup O_2' = X \times (O_1 \sqcup O_2) \prec_{\alpha\times\beta} O \times (X \setminus (O_1 \sqcup O_2))$. Since the two sets in this subequivalence are disjoint and the definitions of O_1' and O_2' did not depend on E, we have verified that $\alpha \times \beta$ is $(O \times X)$-squarely divisible.

By Proposition 2.1, there is a homeomorphism $g : X \times X \to X$ which satisfies $g(O \times X) = O$ and conjugates $\alpha \times \beta$ to an action γ of $G * H$ on X that approximates α as closely as we wish. The first of these conditions together with the $(O \times X)$-square divisibility of $\alpha \times \beta$ ensures that γ is O-squarely divisible. Since $\alpha \times \beta$ is minimal and weakly mixing and admits an invariant Borel probability measure, the action γ has these properties as well. Finally, since α is topologically free, the product $\alpha \times \beta$ is topologically free and hence γ is topologically free. Thus $\gamma \in \mathrm{WA}(G * H, X)$, which finishes the proof. □

Theorem 9.2. *Suppose X is the Cantor set. Suppose that G is residually finite and that H is infinite and amenable and contains a normal infinite cyclic subgroup. Then the set of all weakly squarely divisible actions in* $\mathrm{WA}(G * H, X)$ *is a dense* G_δ.

Proof. Let O be a nonempty clopen subset of X and E a finite subset of $G * H$ containing e. Since there are only countably many such O and E, by the Baire category theorem it suffices to show that the set $\mathscr{W}_{O,E}$ of all (O, E)-squarely divisible actions in $\mathrm{WA}(G * H, X)$ is a dense open set. The openness is Lemma 3.13 and the density is Lemma 9.1 (using that O-square divisibility implies (O, E)-square divisibility). □

For $d \geq 2$ the free group F_d can be written as $G * \mathbb{Z}$ where G is a free product of $d - 1$ copies of $\mathbb{Z}$. From the above theorem we thus obtain:

Corollary 9.3. *For $d \geq 2$ and the Cantor set X the set of all weakly squarely divisible actions in* $\mathrm{WA}(F_d, X)$ *is a dense* G_δ.

Remark 9.4. As mentioned in the introduction, the space of all free minimal actions $F_2 \curvearrowright X$ on the Cantor set with $M_{F_2}(X) \neq \emptyset$ has a generic action, namely the universal odometer [20]. The arguments for establishing Theorem 9.2 also apply to this space and thus show that the universal odometer is weakly squarely divisible, so that its reduced crossed product has stable rank one by Theorem 4.6. One can also verify this directly using translation actions on spaces of the form F_2/H, where H is a finite-index subgroup of F_2, in which one generator acts at a much larger scale than the other so as to effectively reproduce the situation in the proof of Proposition 7.12, but now inside a single clopen tower partitioning the space, with the tower levels indexed by F_2/H and permuted by the translation action of F_2.

10. Generic square divisibility II

For $d > 1$, Corollary 9.3 says that, within the space $\mathrm{WA}(F_d, X)$ of all actions $F_d \curvearrowright X$ that are minimal, weakly mixing, and topologically free and satisfy $M_{F_d}(X) \neq \emptyset$, the weakly squarely divisible ones form a dense G_δ set. We do not know much however about the size of this G_δ set modulo the relation of conjugacy, although we will exhibit many examples of these actions in Section 12.

In this section we examine instead the space $\mathrm{A}^*(F_d, X)$ of topologically free actions $F_d \curvearrowright X$ that satisfy $M_{F_d}(X) \neq \emptyset$ and are strictly ergodic and spectrally aperiodic on each generator. This intersects the space $\mathrm{WA}(F_d, X)$ but neither of these spaces seems to contain the other. As we did for $\mathrm{WA}(F_d, X)$, we will prove that the set of all weakly squarely divisible actions in $\mathrm{A}^*(F_d, X)$ is a dense G_δ (Theorem 10.4), but we will also be able to show that every conjugacy class in $\mathrm{A}^*(F_d, X)$ is meagre (this is Theorem E, established in Section 11), so that the set of actions in $\mathrm{A}^*(F_d, X)$ modulo the relation of conjugacy is quite large from the descriptive viewpoint.

The *continuous discrete spectrum* of a transformation $S \curvearrowright Y$ is the set of eigenvalues of the linear operator $g \mapsto g \circ S$ acting on $C(Y)$.

Lemma 10.1. *Let $S \curvearrowright X$ and $T \curvearrowright Y$ be minimal homeomorphisms. Suppose that there exists a $\nu \in M_T(Y)$ such that $T \curvearrowright (Y, \nu)$ is weakly mixing. Then S and $S \times T$ have the same continuous discrete spectrum.*

Proof. It is clear that the continuous discrete spectrum of S is contained in that of $S \times T$, since (identifying $C(X) \otimes C(Y)$ with $C(X \times Y)$) we can take a tensor product of a continuous eigenfunction for the former with 1_Y to get a continuous eigenfunction for the latter with the same eigenvalue. To establish the reverse inclusion we will show that all continuous eigenfunctions for $S \times T$ arise in this way.

By the amenability of $\mathbb{Z}$ there exists a $\mu \in M_S(X)$. Let $U_S \in \mathcal{B}(L^2(X, \mu))$ and $U_T \in \mathcal{B}(L^2(Y, \nu))$ be the Koopman operators associated to $S \curvearrowright (X, \mu)$ and $T \curvearrowright (Y, \nu)$. Then we have orthogonal decompositions of U_S and U_T as $1 \oplus U_{S,0} \in \mathcal{B}(\mathbb{C} \oplus L^2_0(X, \mu))$ and $1 \oplus U_{T,0} \in \mathcal{B}(\mathbb{C} \oplus L^2_0(Y, \nu))$. The Koopman operator $U_{S\times T}$ associated to the diagonal transformation $S \times T \curvearrowright (X \times Y, \mu \times \nu)$ is then unitarily equivalent to $1 \oplus U_{S,0} \oplus U_{T,0} \oplus (U_{S,0} \otimes U_{T,0})$ acting on $\mathbb{C} \oplus L^2_0(X, \mu) \oplus L^2_0(Y, \nu) \oplus (L^2_0(X, \mu) \otimes L^2_0(Y, \nu))$.

Suppose that $g \in C(X \times Y)$ is an eigenfunction for $S \times T$ with eigenvalue not equal to 1 (which we may assume since the continuous discrete spectra of S and $S \times T$ clearly share the eigenvalue 1 by considering constant functions). Since S and T are minimal the measures μ and ν have full support and hence so does the measure $\mu \times \nu$. We can then view g as an eigenfunction for $U_{S\times T}$ in $L^2(X \times Y, \mu \times \nu)$. Since $T \curvearrowright (Y, \nu)$ is weakly mixing, $U_{T,0}$ is weakly mixing as a unitary operator (see [45, Theorem 2.25]), which also means that the operator $U_{S,0} \otimes U_{T,0}$ is weakly mixing (see [45, Theorem 2.23]). Since weakly mixing unitary operators have no eigenvalues and g is associated with an eigenvalue different from 1, this implies that g belongs to $L^2_0(X, \mu) \otimes \mathbb{C}1_Y$ and thus, modulo a.e. equality, has the form $f \otimes 1_Y$ for some $f \in L^2_0(X, \mu)$. It follows that for μ-a.e. $x \in X$ we have $g(x, y) = f(x)$ for ν-a.e. $y \in Y$. The Y-cross sections of g, namely $g(x, \cdot) : Y \to \mathbb{C}$ for μ-a.e. $x \in X$, are therefore ν-a.e. constant and thus constant by the continuity of g and the fact that ν has full support. Now the set $\{x \in X : g(x, y) = f(x) \text{ for all } y \in Y\}$ must be dense in X since μ has full support. We deduce, again using the continuity of g, that g has the form $g_0 \otimes 1_Y$ for some $g_0 \in C(X)$. But then g_0 is a continuous eigenfunction for S with the same eigenvalue that g has as an eigenfunction for $S \times T$. We conclude that the continuous discrete spectrum of $S \times T$ is contained in that of S, as desired. □

Lemma 10.2. *Suppose X is the Cantor set. Let $S \curvearrowright X$ and $T \curvearrowright X$ be homeomorphisms such that S is spectrally aperiodic and there exists a $\nu \in M_T(X)$ for which $T \curvearrowright (X, \nu)$ is weakly mixing. Suppose that $S \times T$ is minimal. Then $S \times T$ is spectrally aperiodic.*

Proof. Since $S \times T \curvearrowright X \times X$ is minimal, the factors S and T are themselves minimal. As observed in [61] (see also [30]), the minimal homeomorphism S is spectrally aperiodic if and only if there does not exist a positive integer $n \neq 1$ such that $e^{2\pi i/n}$ belongs to the continuous discrete spectrum of S, and likewise for $S \times T$. Since the continuous discrete spectra of S and $S \times T$ agree by Lemma 10.1, it follows from our assumption on S that $S \times T$ is spectrally aperiodic. □

Lemma 10.3. *Suppose X is the Cantor set. Let $d \in \mathbb{N}$ and let $O \subseteq X$ be a nonempty clopen set. Then the set of O-squarely divisible actions in $\mathrm{A}^*(F_d, X)$ is dense.*

Proof. Let $\alpha \in \mathrm{A}^*(F_d, X)$ and $\mu \in M_\alpha(X)$. Let $T_1, \dots, T_d$ be the homeomorphisms obtained by restricting α to the d standard generators of F_d. For each $i = 1, \dots, d$ write μ_i for the unique measure in $M_{T_i}(X)$, which is ergodic. Since each T_i is minimal and X is infinite, the measures μ_i must be atomless. Therefore there exists a standard atomless Borel probability space (Z, ζ) such that $(X, \mu_i) \cong (Z, \zeta)$ for every $i = 1, \dots, d$. By Lemma 7.8 there is a weakly mixing p.m.p. transformation $S \curvearrowright (Z, \zeta)$ which is disjoint from $T_i \curvearrowright (X, \mu_i)$ for each $i = 1, \dots, d$. By the Jewett–Krieger theorem there is a minimal homeomorphism $R \curvearrowright X$ such that $M_R(X)$ contains a unique element ν and the transformations $R \curvearrowright (X, \nu)$ and $S \curvearrowright (Z, \zeta)$ are measurably conjugate. By Lemma 7.2 this implies that for each $i = 1, \dots, d$ the diagonal transformation $T_i \times R \curvearrowright X \times X$ is minimal and, by Lemma 10.2, spectrally aperiodic. It is also uniquely ergodic by the unique ergodicity of T_i and R and the disjointness of $T_i \curvearrowright (X, \mu_i)$ and $R \curvearrowright (X, \nu)$.

By the minimality of T_1 and the compactness of X we can find a clopen partition $\{W_k\}_{k=1}^l$ of X and distinct integers $j_1, \dots, j_l \in \mathbb{Z}$ such that $T_1^{j_k} W_k \subseteq O$ for every $k = 1, \dots, l$. We moreover may assume that $j_1, \dots, j_l \in \mathbb{Z} \setminus \{0\}$ since T_1 is minimal and X has no isolated points. Define an action $\beta \in \mathrm{Act}(F_d, X)$ by setting $\beta_{a_i} = R$ for every $i = 1, \dots, d$, where $a_1, \dots, a_d$ are the standard generators of F_d. Since X is infinite, the homeomorphism R, being minimal, is free as a $\mathbb{Z}$-action, and so we can find a nonempty clopen set $A \subseteq X$ such that the sets $R^{j_k} A$ for $k = 1, \dots, l$ are pairwise disjoint and also disjoint from A. Fix O_1 and O_2 nonempty clopen disjoint subsets of A (which exist since X has no isolated points). As in the proof of Lemma 9.1, we then have

$$
\begin{aligned}
X \times (O_1 \sqcup O_2) &= \bigsqcup_{k=1}^{l} W_k \times (O_1 \sqcup O_2) \\
&\prec_{T_1 \times R} \bigsqcup_{k=1}^{l} T_1^{j_k} W_k \times R^{j_k}(O_1 \sqcup O_2) \\
&\subseteq O \times (X \setminus (O_1 \sqcup O_2)).
\end{aligned}
$$

By Theorem 5.7 the action R is (O_1, O_2, J)-squarely divisible for all finite sets $J \subseteq \mathbb{Z}$ containing 0. Arguing as in the proof of Lemma 9.1, this shows that $\alpha \times \beta$ is $(X \times O_1, X \times O_2, E)$-squarely divisible for every finite set $E \subseteq F_d$ containing e (here we use the fact that, by the definition of β, for every $g \in F_d$ there exists a unique $N_g \in \mathbb{Z}$ for which $\beta_g = R^{N_g}$). Moreover, since the two sets in the subequivalence $X \times (O_1 \sqcup O_2) \prec_{T_1 \times R} O \times (X \setminus (O_1 \sqcup O_2))$ are pairwise disjoint, we conclude that $\alpha \times \beta$ is $(O \times X)$-squarely divisible (see Proposition 3.9). Again as in the proof of Lemma 9.1, we can conjugate $\alpha \times \beta$ to an action γ on X via a homeomorphism $g : X \times X \to X$

satisfying $g(O \times X) = O$ so that γ approximates α as closely as we wish. In view of the $(O \times X)$-square divisibility of $\alpha \times \beta$, the equality $g(O \times X) = O$ ensures that γ is O-squarely divisible. Since α is topologically free, so is $\alpha \times \beta$. Moreover $M_{\alpha\times\beta}(X \times X) = \{\mu \times \nu\}$, while $\alpha \times \beta$ is strictly ergodic and spectrally aperiodic on each standard generator of F_d. Therefore $\gamma \in \mathrm{A}^*(F_d, X)$, which completes the proof. □

Theorem 10.4. *Suppose X is the Cantor set. For $d \in \mathbb{N}$ the set of all weakly squarely divisible actions in $\mathrm{A}^*(F_d, X)$ is a dense G_δ.*

Proof. As in the proof of Theorem 9.2, it suffices to show that for every nonempty clopen set $O \subseteq X$ and finite set $e \in E \subseteq F_d$ the set $\mathscr{W}_{O,E}$ of all (O, E)-squarely divisible actions in $\mathrm{A}^*(F_d, X)$ is open and dense. Lemma 3.13 yields the openness and Lemma 10.3 yields the density (using that O-square divisibility implies (O, E)-square divisibility). □

11. Meagreness of orbits: proof of Theorem E

Our goal here is to establish the meagreness-of-orbits result announced in the introduction as Theorem E. For this we develop a topological version of the entropy-and-disjointness argument from Section 4 of [25] that involved extending the generic disjointness result for ergodic p.m.p. $\mathbb{Z}$-actions from [17] (whose proof did not involve entropy) to the more general setting of amenable groups.

Actually our disjointness argument concerns $\mathbb{Z}$-actions that arise as restrictions of actions $F_d \cong \mathbb{Z} * F_{d-1}$ preserving a Borel probability measure. In [40], and like in [17] without the use of entropy, Hochman establishes a generic disjointness result in the space of homeomorphisms of the Cantor set that we would be able to apply here if it were not for the constraint that we cannot simply perturb the $\mathbb{Z}$ part of the action at will. Such a perturbation will always generate an action of the free product $\mathbb{Z} * F_{d-1} \cong F_d$ but can easily result in the nonexistence of Borel probability measures that are invariant for the whole F_d. Our strategy has been to instead combine the entropy approach of [25] with the orbit equivalence theory of Cantor minimal systems.

Notation 11.1. Suppose that X is the Cantor set. We denote by $\mathrm{A}^*(\mathbb{Z}, X)$ the space of strictly ergodic spectrally aperiodic actions in $\mathrm{Act}(\mathbb{Z}, X)$ (this agrees with $\mathrm{A}^*(F_d, X)$, as defined in Section 8, when $d = 1$). For $T \in \mathrm{A}^*(\mathbb{Z}, X)$ we write $\mathrm{A}^*(\mathbb{Z}, X)_T$ for the set of all transformations in $\mathrm{A}^*(\mathbb{Z}, X)$ which have the same orbits as T.

When dealing with homeomorphisms $T \curvearrowright X$ of the Cantor set in this and the next section, our clopen castles will always partition X (i.e., have empty remainder) and have shapes of the form $\{0, \dots, n-1\}$, even when this is not explicitly stated. We write $\mathscr{C} = \{(B_i, n_i)\}_{i=1}^k$ for such a castle when the shape of the ith tower is the interval $\{0, \dots, n_i - 1\}$. Note that if $p \in \mathbb{N}$ divides all of the heights n_i, then one can find levels of the castle whose union A produces a partition $X = A \sqcup TA \sqcup \cdots \sqcup T^{p-1}A$. Consequently, if T is spectrally aperiodic then $\gcd(n_1, n_2, \dots, n_k) = 1$. Write $\mathrm{base}(\mathscr{C}) = \bigsqcup_{i=1}^k B_i$ and $\mathrm{top}(\mathscr{C}) = \bigsqcup_{i=1}^k T^{n_i-1}B_i$, and note that $T(\mathrm{top}(\mathscr{C})) = \mathrm{base}(\mathscr{C})$. We will moreover consider measurable castles with interval shapes for p.m.p. transformations $S \curvearrowright (Z, \zeta)$, where the sets B_i are instead required to be measurable and we similarly ask that the collection $\{S^j B_i : j = 0, \dots, n_i - 1, i = 1, \dots, k\}$ is disjoint and has union of full measure.

Given a minimal homeomorphism $T \curvearrowright X$ of the Cantor set, by a standard first return time construction one can generate a clopen castle that partitions X starting from any nonempty clopen set $Y \subseteq X$. Indeed a straightforward consequence of compactness and minimality is the existence of an $N \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{N} T^{-n}Y$. The first return map $r_Y : Y \to \mathbb{N}$ determined by $r_Y(x) = \min\{n \in \mathbb{N} : T^n x \in Y\}$ for $x \in Y$ is then well-defined and continuous (since Y is clopen) and takes finitely many values $\{n_1, \dots, n_k\}$. Setting $B_i = r_Y^{-1}(\{n_i\})$ for $i = 1, \dots, k$, we obtain a clopen castle decomposition

$$X = \bigsqcup_{i=1}^{k} \bigsqcup_{j=0}^{n_i-1} T^j B_i.$$

Note that the base of this castle is exactly Y.

The following theorem can be extracted from [61]. The fact that the set $\mathscr{M}$ below is nonempty is a special case of Theorem 6.1 of [61], and the fact that it intersects every neighbourhood of T can be verified by making a careful choice of the initial castle in the recursive construction in [61], as we explain below. Recall that two free transformations $T \curvearrowright X$ and $S \curvearrowright Y$ are said to be *strongly orbit equivalent* if there exists an orbit-preserving homeomorphism $\varphi : X \to Y$ between them with associated cocycle maps (as defined in Section 3) that have no more than one point of discontinuity each.

Theorem 11.2. [61, Theorem 6.1] *Let $T \curvearrowright X$ be a minimal spectrally aperiodic homeomorphism of the Cantor set and let $\mu \in M_T(X)$ be an ergodic measure. Let $S \curvearrowright (Z, \zeta)$ be an ergodic p.m.p. transformation of a standard atomless probability space. Let $\mathscr{M}$ be the set of all minimal homeomorphisms $T' \curvearrowright X$ such that the identity map $\mathrm{id} : X \to X$ is a strong orbit equivalence between T and T' and $T' \curvearrowright (X, \mu)$ is measure conjugate to $S \curvearrowright (Z, \zeta)$ (note that $M_{T'}(X) = M_T(X)$ since $\mathrm{orb}_T(x) = \mathrm{orb}_{T'}(x)$ for every $x \in X$). Then every neighbourhood of T in $\mathrm{Act}(\mathbb{Z}, X)$ contains an element of $\mathscr{M}$.*

We first sketch the ideas in [38] and [61] that yield $\mathscr{M} \neq \emptyset$. For more on Bratteli diagrams, orderings on such diagrams, and Bratteli–Vershik transformations, we refer the reader to [66]. Ormes proves that there exists a simple Bratteli diagram (V, E) with two orderings $\leq$ and $\leq'$ such that

(i) $(V, E, \leq)$ and $(V, E, \leq')$ share minimal and maximal edges and both have unique minimal and maximal paths,
(ii) $(V, E, \leq)$ defines a Bratteli–Vershik homeomorphism λ of $B_{V,E}$ and there is a homeomorphism $h : X \to B_{V,E}$ such that $T = h^{-1} \circ \lambda \circ h$,
(iii) $(V, E, \leq')$ defines a Bratteli–Vershik homeomorphism λ' of $B_{V,E}$ such that $S \curvearrowright (Z, \zeta)$ is measure conjugate to $\lambda' \curvearrowright (B_{V,E}, h_*\mu)$.

One can then check that λ and λ' are strongly orbit equivalent via the identity map and therefore $T' := h^{-1} \circ \lambda' \circ h$ belongs to $\mathscr{M}$.

In what follows we explain how the above Bratteli diagrams arise, following [38, 61]. See Figures 2 and 3 for an illustration. Suppose that $\mathscr{C}_1 \prec \mathscr{C}_2 \prec \mathscr{C}_3 \prec \dots$ is a sequence of clopen castles for T, where $\mathscr{C}_m \prec \mathscr{C}_{m+1}$ means that the levels of the castle $\mathscr{C}_m$ are unions of levels of the castle $\mathscr{C}_{m+1}$ and $\mathrm{base}(\mathscr{C}_{m+1}) \subseteq \mathrm{base}(\mathscr{C}_m)$. Let $\mathscr{C}_0 = \{(X, 0)\}$ be the trivial castle. Such a sequence gives rise to a Bratteli diagram (V, E) with vertex sets $\{V_m\}_{m \geq 0}$ and edge sets $\{E_m\}_{m \geq 1}$ defined as follows: the vertices in V_m are indexed by the towers in $\mathscr{C}_m$, so that $|V_m|$ is the number of towers forming the castle $\mathscr{C}_m$. The number of edges $e \in E_m$ going from the vertex $k \in V_{m-1}$ to the vertex $l \in V_m$ is defined to be the number of levels of the lth tower of $\mathscr{C}_m$

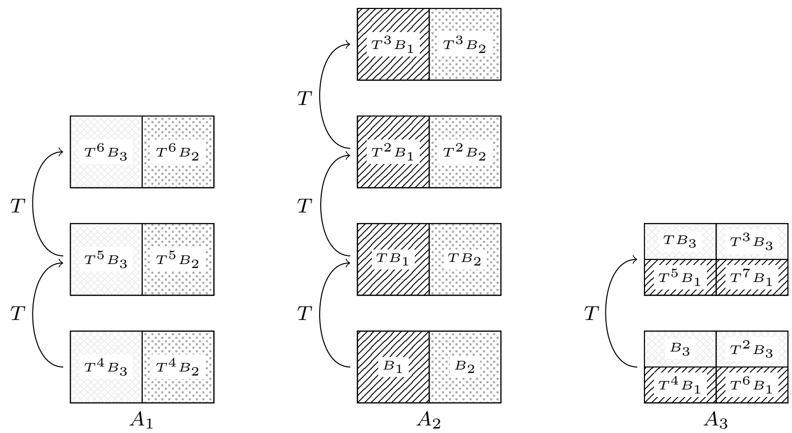

FIGURE 2. Castles $\mathscr{C}_1 \prec \mathscr{C}_2$ for T with $\text{base}(\mathscr{C}_1) = A_1 \sqcup A_2 \sqcup A_3$ and $\text{base}(\mathscr{C}_2) = B_1 \sqcup B_2 \sqcup B_3$.

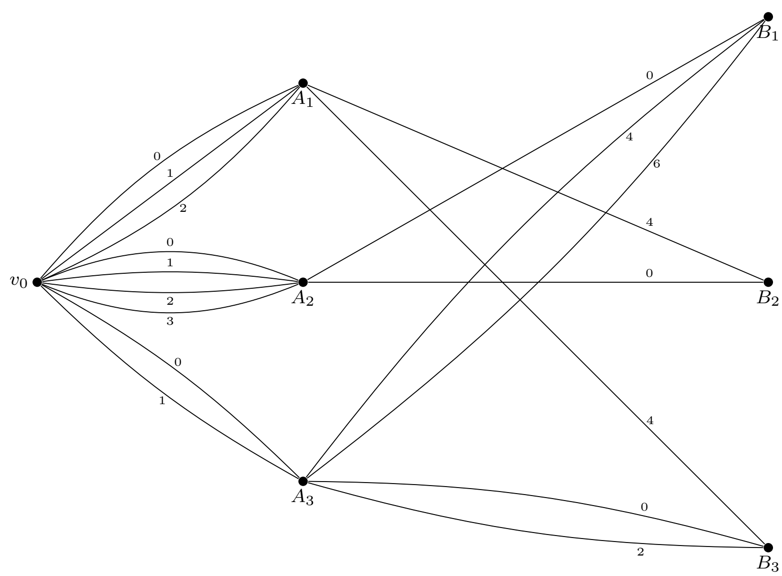

FIGURE 3. The initial paths of length two in the ordered Bratteli diagram associated to Figure 2.

which are contained in the base of the kth tower of $\mathscr{C}_{m-1}$. Note for example that the number of edges going from the only vertex $1 \in V_0$ to the vertex $i \in V_1$ is equal to the height of the ith tower of the castle $\mathscr{C}_1$.

Next define an ordering $\leq$ on the Bratteli diagram as follows. For each vertex $l \in V_m$ one introduces a total ordering on the subset $r^{-1}(\{l\}) \subseteq E_m$, where $r : E_m \to V_m$ denotes the range map. Writing $\mathscr{C}_m = \{(B_i, n_i)\}_{i=1}^k$, we note by construction that the edges in $r^{-1}(\{l\})$ can be naturally labeled by the set of integers $\{j \in \{0, \dots, n_l - 1\} : T^j B_l \subseteq \text{base}(\mathscr{C}_{m-1})\}$. For instance, if $l \in V_1$ then $r^{-1}(\{l\})$ is labeled by the entire interval $\{0, \dots, n_l - 1\}$. This labeling induces a natural ordering on $r^{-1}(\{l\})$ via the standard order on the integers. One moreover shows that for every $m \in \mathbb{N}$ there is a bijection

$$\{\text{initial paths of length } m \text{ in } (V,E)\} \to \{\text{levels of the castle } \mathscr{C}_m\}$$

defined by $e_1 e_2 \cdots e_m \mapsto T^{\sum_{j=1}^m \mathrm{label}(e_j)} B_{m,r(e_m)}$, where $B_{m,r(e_m)}$ is the base of the $r(e_m)$th tower of the castle $\mathscr{C}_m$. If $\bigvee_{m\in\mathbb{N}} \mathscr{C}_m$ generates the topology on X, then the compatibility of these maps leads to a homeomorphism $B_{V,E} \cong X$ mapping an infinite path $e_1 e_2 \cdots \in B_{V,E}$ to $\bigcap_{m=1}^{\infty} T^{\sum_{j=1}^m \mathrm{label}(e_j)} B_{m,r(e_m)}$, which is an intersection of a decreasing sequence of sets. Denote the inverse of this homeomorphism by $h : X \to B_{V,E}$. If $\bigcap_{m\in\mathbb{N}} \mathrm{top}(\mathscr{C}_m) = \{x_0\}$ then the Bratteli diagram constructed in this way has unique minimal and maximal paths $x_{\min}$ and $x_{\max}$ with $h(x_0) = x_{\max}$, and it is straightforward to check, by the definition of the ordering, that the Bratteli-Vershik homeomorphism λ of $B_{V,E}$ satisfies $h \circ T = \lambda \circ h$.

This is essentially how condition (ii) above is arranged. However, in order to achieve a Bratteli diagram satisfying conditions (i)–(iii), a recursive procedure is used to simultaneously build a sequence of measurable castles $\mathscr{D}_1 \prec \mathscr{D}_2 \prec \mathscr{D}_3 \prec \dots$ for S (with the relation $\prec$ defined on measurable castles in the same way as for clopen castles). Letting again $\mathscr{D}_0$ be the trivial castle, we obtain, exactly as above, an associated ordered Bratteli diagram $(V', E', \leq')$, and if $\{\mathscr{D}_m\}_{m\in\mathbb{N}}$ separates points on a set of full ζ-measure, one additionally gets a measurable isomorphism $h' : (Z,\zeta) \to (B_{V',E'}, h'_*\zeta)$ that conjugates S to the Bratteli–Vershik homeomorphism λ' of $B_{V',E'}$, provided that $(V', E', \leq')$ has unique minimal and maximal paths.

In this recursive construction, the initial clopen castle $\mathscr{C}_1$ can be chosen freely. During the simultaneous construction of the sequences $\mathscr{C}_1 \prec \mathscr{C}_2 \prec \dots$ and $\mathscr{D}_1 \prec \mathscr{D}_2 \prec \dots$ one requires that $\mathscr{C}_m$ and $\mathscr{D}_m$ have the same number of towers with the same corresponding heights for all $m \in \mathbb{N}$. This ensures that V can be identified with V'. A further technical ad hoc compatibility condition is imposed on the levels of the castles to guarantee that $E = E'$ and that the minimal and maximal edges coincide for the orderings $\leq$ and $\leq'$ on (V,E). Finally, one requires that if $\mathscr{C}_m = \{(B_i, n_i)\}_{i=1}^k$ and $\mathscr{D}_m = \{(A_i, n_i)\}_{i=1}^k$ then $\mu(B_i) = \zeta(A_i)$ for all $i = 1, \dots, k$. This condition ensures that the measures $h'_*\zeta$ and $h_*\mu$ on $B_{V,E}$ agree, so that the map $h' : (Z,\zeta) \to (B_{V,E}, h_*\mu)$ defined above is a measurable isomorphism conjugating S to λ' as required in condition (iii).

A key tool in this construction is Alpern's copying lemma (see [2, Corollary 2] and [61, Theorem 4.1]), which in our case can be applied in its original form since T is assumed to be spectrally aperiodic (which, as explained earlier in this section, implies that each associated castle $\mathscr{C}_m$ has towers whose heights have greatest common divisor equal to 1). Using Alpern's lemma, one can, for any clopen castle $\mathscr{C}$ of T, construct a measurable castle $\mathscr{D}$ for S with the same number of towers and identical corresponding heights such that the μ-measures of the levels of $\mathscr{C}$ agree with the ν-measures of the corresponding levels of $\mathscr{D}$. In particular, Alpern's lemma is already used in the first step to construct $\mathscr{D}_1$.

Given a neighbourhood U of T in $\mathrm{Act}(\mathbb{Z}, X)$, we can construct a $\mathscr{C}_1$ as follows in order to verify that $T' = h^{-1} \circ \lambda' \circ h$ moreover belongs to U.

Let $\varepsilon > 0$ be such that every $\tilde{T} \in \mathrm{Act}(\mathbb{Z}, X)$ with $d(\tilde{T}x, Tx), d(\tilde{T}^{-1}x, T^{-1}x) < \varepsilon$ for all $x \in X$ belongs to U. Take a nonempty clopen set $B \subseteq X$ with diameter less than ε such that $T^{-1}B$ also has diameter less than ε (which is possible by the continuity of T^{-1}). As described at the beginning of this section, we then obtain a clopen castle $\mathscr{C}_1 = \{(B_i, n_i)\}_{i=1}^k$ with $\mathrm{base}(\mathscr{C}_1) = B$, via first-return times to B. We may assume that the levels of $\mathscr{C}_1$ all have diameter less than ε. Indeed, let $\mathscr{P}$ be any clopen partition of X with elements of diameter less than ε and partition each B_i by $\bigvee_{j=0}^{n_i-1} T^{-j}\mathscr{P}$ so as to obtain a clopen partition $B_{i,0},\dots,B_{i,r_i}$ of B_i. The castle $\{\{(B_{i,j},n_i)\}_{j=0}^{r_i}\}_{i=1}^{k}$ has base equal to B and levels of diameter less than ε.

Let A be a level of the castle $\mathscr{C}_1$ that is not at the top of a tower, i.e., $A = T^j(B_i)$ for some $1 \le i \le k$ and $0 \le j \le n_i - 2$. We claim that $T'(A) = h^{-1}\circ\lambda'\circ h(A) = h^{-1}\circ\lambda\circ h(A) = T(A)$. Note that $h(A)$ is the set

$$\{e_1e_2\cdots \in B_{V,E} : r(e_1) \text{ corresponds to the } i\text{th tower of } \mathscr{C}_1 \text{ and label}(e_1) = j\}.$$

By the construction above, the edges that join the vertex in V_1 corresponding to the ith tower of $\mathscr{C}_1$ to the unique vertex in the level V_0 are ordered in the same way by $\le$ and $\le'$ (as induced from the natural order on the interval $\{0,1,\dots,n_i-1\}$), so that the images of $h(A)$ under λ and λ' are identical and equal to

$$\{e_1e_2\cdots \in B_{V,E} : r(e_1) \text{ corresponds to the } i\text{th tower of } \mathscr{C}_1 \text{ and label}(e_1) = j+1\}.$$

We conclude that $T'(A) = T(A)$.

Assume now that $A = \mathrm{top}(\mathscr{C}_1)$. We will show that $T'(A) = T(A)$. Note that

$$h(A) = \{e_1e_2\cdots \in B_{V,E} : e_1 \text{ is a maximal edge (w.r.t. both orders)}\}.$$

We let $\mathfrak{s}$ and $\mathfrak{p}$ denote the successor and predecessor operations, with respect to $\le$, on edges in the totally ordered sets $r^{-1}(v)$ for $v \in V$. The analogous operations associated with the order $\le'$ we denote by $\mathfrak{s}'$ and $\mathfrak{p}'$. We want to show that $\lambda(h(A)) = \lambda'(h(A))$. Clearly both sets contain the unique (shared) minimal path $x_{\min}$ in $B_{V,E}$, since the unique (shared) maximal path $x_{\max}$ belongs to $h(A)$. Assume now that $e_1e_2\cdots \in h(A)$ is not the maximal path, and let $n \ge 2$ be the first integer such that $e_n \in E_n$ is not a maximal edge (w.r.t. both orders). Then $\lambda(e_1e_2\cdots) = y_1\cdots y_{n-1}\mathfrak{s}(e_n)e_{n+1}\cdots$ where $y_1\cdots y_{n-1}$ is the unique path consisting of minimal edges (w.r.t. both orders) such that the concatenation $y_{n-1}\mathfrak{s}(e_n)$ is well-defined. We need to show that $y_1\cdots y_{n-1}\mathfrak{s}(e_n)e_{n+1}\cdots$ belongs to $\lambda'(h(A))$. Note that since $\le$ and $\le'$ share minimal and maximal edges, $\mathfrak{s}(e_n)$ cannot be minimal for $\le'$, and so $\mathfrak{p}'(\mathfrak{s}(e_n))$ is well-defined. Let $q = z_1\cdots z_{n-1}\mathfrak{p}'(\mathfrak{s}(e_n))e_{n+1}\cdots$ where $z_1\cdots z_{n-1}$ is the unique path consisting of maximal edges (w.r.t. both orders) such that the concatenation $z_{n-1}\mathfrak{p}'(\mathfrak{s}(e_n))$ is well-defined. Clearly, $q \in h(A)$, and we have $\lambda'(q) = y_1\cdots y_{n-1}\mathfrak{s}(e_n)e_{n+1}\cdots$, completing the argument. Since $A = T^{-1}B$, we obtain $T'(T^{-1}B) = T(T^{-1}B) = B$ and finally conclude that $d(T'x,Tx) < \varepsilon$ for all $x \in X$.

To see that $d(T'^{-1}x, T^{-1}x) < \varepsilon$ for all $x \in X$, one similarly shows that for every level A that is not at the base of $\mathscr{C}_1$ one has $T'^{-1}A = T^{-1}A$, while $T'^{-1}B = T^{-1}B$. It follows by our choice of ε that T' belongs to U. This establishes the conclusion of Theorem 11.2.

Lemma 11.3. *Suppose X is the Cantor set. Let $T \in \mathrm{A}^*(\mathbb{Z},X)$ and let U be a neighbourhood of T in $\mathrm{Act}(\mathbb{Z},X)$. Then U contains an action in $\mathrm{A}^*(\mathbb{Z},X)_T$ with uniformly positive entropy.*

Proof. Since T belongs to $\mathrm{A}^*(\mathbb{Z},X)$ there is a unique measure μ in $M_T(X)$. By Theorem 11.2 we can find a minimal homeomorphism $T' \in U$ with the same orbits as T such that $T' \curvearrowright (X,\mu)$ is a Bernoulli shift. The fact that $T' \curvearrowright (X,\mu)$ is Bernoulli implies that it is mixing. By minimality μ has full support, and so $T' \curvearrowright X$ is topologically mixing and hence spectrally aperiodic, so that T' belongs to $\mathrm{A}^*(\mathbb{Z},X)_T$. Bernoullicity also implies by Theorem A of [34] that T' has uniformly positive entropy. □

Lemma 11.4. *Suppose X is the Cantor set. Let $T \in \mathrm{A}^*(\mathbb{Z},X)$ and let U be a neighbourhood of T in $\mathrm{Act}(\mathbb{Z},X)$. Then U contains an action in $\mathrm{A}^*(\mathbb{Z},X)_T$ with zero entropy.*

Proof. We follow the ideas in the proof of [61, Corollary 6.3]. Write μ for the unique measure in $M_T(X)$. Let $S \curvearrowright (Z, \zeta)$ be a weakly mixing p.m.p. transformation on an atomless standard probability space with zero measure entropy. The existence of such an S follows, for example, from the genericity of weak mixing [36] and zero measure entropy [70] in $\mathrm{Aut}(Z, \zeta)$. By Theorem 11.2, there exists a minimal homeomorphism $T' \in U$ such that $T' \curvearrowright X$ is strongly orbit equivalent to $T \curvearrowright X$ via the identity map and $T' \curvearrowright (X, \mu)$ is measure conjugate to $S \curvearrowright (Z, \zeta)$. Since μ has full support by minimality, we conclude that $T' \curvearrowright X$ is topologically weakly mixing, and in particular spectrally aperiodic. Finally, since T' is uniquely ergodic and $T' \curvearrowright (X, \mu)$ has zero measure entropy, it follows by the variational principle [45, Theorem 9.48] that $T' \curvearrowright X$ has zero topological entropy. □

Let $d \geq 2$. Recall that $\mathrm{A}^*(F_d, X)$ denotes the set of all topologically free actions $\alpha \in \mathrm{Act}(F_d, X)$ with $M_\alpha(X) \neq \emptyset$ that are spectrally aperiodic and strictly ergodic on each standard generator (conditions which imply that α itself is strictly ergodic).

Lemma 11.5. *Let X be the Cantor set and $\gamma \in \mathrm{A}^*(F_d, X)$. Fix one of the standard generators a of F_d. Then the set*

$$\mathscr{D}_\gamma := \{\beta \in \mathrm{A}^*(F_d, X) : \text{the homeomorphisms } \beta_a \text{ and } \gamma_a \text{ are disjoint}\}$$

is a G_δ.

Proof. Since the map $\gamma \mapsto \gamma_a$ from $\mathrm{A}^*(F_d, X)$ to $\mathrm{Act}(\mathbb{Z}, X)$ is clearly continuous and its image is contained in the set of minimal actions in $\mathrm{Act}(\mathbb{Z}, X)$, it suffices to show, given a minimal homeomorphism $T \in \mathrm{Act}(\mathbb{Z}, X)$, that the set $T^\perp$ of all minimal homeomorphisms of X which are disjoint from T is a G_δ. Let U be a nonempty open subset of $X \times X$. Write $\mathscr{V}_U$ for the set of all $S \in \mathrm{Act}(\mathbb{Z}, X)$ such that every orbit of $T \times S$ intersects U. Let us verify that $\mathscr{V}_U$ is open. Let $S \in \mathscr{V}_U$. Then for every $z \in X \times X$ there is an $n_z \in \mathbb{Z}$ such that $(T \times S)^{n_z} z \in U$ and hence by continuity an open neighbourhood V_z containing z such that $(T \times S)^{n_z} V_z \subseteq U$. By the compactness of $X \times X$ there is a finite set $E \subseteq X \times X$ such that the sets V_z for $z \in E$ cover $X \times X$. For all S' in a sufficiently small neighbourhood of S in $\mathrm{Act}(\mathbb{Z}, X)$ we have $(T \times S')^{n_z} V_z \subseteq U$ for every $z \in E$, in which case $S' \in \mathscr{V}_U$. Thus $\mathscr{V}_U$ is open.

By metrizability there is a sequence $U_1, U_2, \dots$ of nonempty open subsets of $X \times X$ that form a basis for the topology. Then, using the fact that the disjointness of two minimal homeomorphisms is equivalent to the minimality of the diagonal action, one sees that the set $T^\perp$ is equal to the set of minimal actions in $\mathrm{Act}(\mathbb{Z}, X)$ intersected with $\bigcap_{n \in \mathbb{N}} \mathscr{V}_{U_n}$ and hence is a G_δ. □

Proof of Theorem E. Suppose, towards a contradiction, that there is a $\gamma \in \mathrm{A}^*(F_d, X)$ whose conjugacy class is nonmeagre. Fix a standard generator $a \in F_d$. Since the map $\rho \mapsto \rho_a$ from $\mathrm{A}^*(F_d, X)$ to $\mathrm{Act}(\mathbb{Z}, X)$ is continuous and the set of homeomorphisms in $\mathrm{Act}(\mathbb{Z}, X)$ with zero entropy is a G_δ [35, Lemma 2.4], the set

$$Z = \{\rho \in \mathrm{A}^*(F_d, X) : \rho_a \text{ has zero entropy}\}$$

is a G_δ.

To see that Z is dense in $\mathrm{A}^*(F_d, X)$, we start by letting $\alpha \in \mathrm{A}^*(F_d, X)$. Then there is a unique $\mu \in M_\alpha(X)$. Since α_a is strictly ergodic by the definition of $\mathrm{A}^*(F_d, X)$, the measure μ is also the unique element of $M_{\alpha_a}(X)$ and thus is ergodic for α_a. Lemma 11.4 tells us that we can find a $T \in \mathrm{A}^*(\mathbb{Z}, X)_{\alpha_a}$ with zero entropy that is as close as we wish to α_a. Consider the action α' of F_d obtained by sending $a \mapsto T$ and having α' equal α on the other standard generators of F_d. Then α' is strictly ergodic and spectrally aperiodic on each of the standard generators of F_d. Since T has the same orbits as of α_a, the measure μ is also T-invariant, whence $M_{\alpha'}(X)$ contains μ and in particular is nonempty. With the aim of showing that α' belongs to $\mathrm{A}^*(F_d, X)$, it remains to verify topological freeness.

Since α is topologically free by virtue of its membership in $\mathrm{A}^*(F_d, X)$, there exists a dense set $X_0 \subseteq X$ such that $\alpha_t x \neq x$ for all $t \in F_d \setminus \{e\}$ and $x \in X_0$. Let $s \in F_d \setminus \{e\}$. As an element of $\langle a \rangle * F_{d-1} = F_d$ we can write s uniquely as a reduced word $a^{n_1} g_1 a^{n_2} g_2 \cdots a^{n_k} g_k$ where possibly $n_1 = 0$ and/or $g_k = e$ (but not both when $k = 1$) but otherwise $n_i \neq 0$ and $g_i \neq e$ for all i. Since T and the restriction of α to $\langle a \rangle$ have the same orbits, for every $x \in X$ we can write $\alpha'_s x$ as $\alpha_t x$ where $t = a^{m_1} g_1 a^{m_2} g_2 \cdots a^{m_k} g_k$ for some $m_1, \dots, m_k \in \mathbb{Z}$ which depend on x and are nonzero (using the freeness of α_a) except in the case when $n_1 = 0$. In particular this t will be nonzero, and so when $x \in X_0$ we get $\alpha'_s x = \alpha_t x \neq x$, from which we conclude that α' is topologically free and hence belongs to $\mathrm{A}^*(F_d, X)$.

The action α' we can make as close as we wish to α, depending on how close T is to α_a. This shows that Z is dense in $\mathrm{A}^*(F_d, X)$, and hence is a dense G_δ. The conjugacy class of γ, being nonmeagre, must then intersect Z, so that γ_a has zero entropy by the invariance of entropy under conjugacy.

Applying Lemma 11.3 in the same way that we did with Lemma 11.4 above, we see that the set of all $\rho \in \mathrm{A}^*(F_d, X)$ such that ρ_a has uniformly positive entropy is dense in $\mathrm{A}^*(F_d, X)$. By Proposition 6 of [8], every minimal homeomorphism of X with zero entropy is disjoint from every homeomorphism of X with uniformly positive entropy, and so we deduce using Lemma 11.5 that $\mathscr{D}_\gamma = \{\beta \in \mathrm{A}^*(F_d, X) : \beta_a \text{ and } \gamma_a \text{ are disjoint}\}$ is a dense G_δ subset of $\mathrm{A}^*(F_d, X)$. But then the conjugacy class of γ, being nonmeagre, must intersect $\mathscr{D}_\gamma$, contradicting the fact that a homeomorphism of any compact metrizable space with more than one point is not disjoint from itself. □

12. Examples of squarely divisible actions of free groups

The purpose of this section is to establish Propositions 12.6 and 12.13, which together provide examples of topologically free actions $F_2 \curvearrowright X$ on the Cantor set with $M_{F_2}(X) \neq \emptyset$ that are (i) strictly ergodic and weakly mixing on each generator and (ii) squarely divisible. These actions in particular belong to both $\mathrm{WA}(F_2, X)$ and $\mathrm{A}^*(F_2, X)$. By Theorem 4.6, their reduced crossed products have stable rank one. One can also similarly construct actions $F_d \curvearrowright X$ for $d \in \mathbb{N}$, $d > 2$, with the same properties.

Given a minimal homeomorphism $T \curvearrowright X$ of the Cantor set and any nonempty clopen set $Y \subseteq X$, by the first return time construction described in Section 11 one can generate a clopen castle $\mathscr{C} = \{(B_i, n_i)\}_{i=1}^k$ that partitions X, with $\mathrm{base}(\mathscr{C}) = Y$. The $\mathbb{Z}$-action generated by T is free by minimality and the fact that X is infinite, and so we can arrange for the heights n_i of the towers to be as large as we wish by choosing Y so that its images under a sufficiently large number of iterations of T^{-1} are pairwise disjoint.

To construct our actions $F_2 = \langle a, b \rangle \curvearrowright X$ on the Cantor set, we will take a minimal homeomorphism T of X (corresponding to a) and then combine this with a second homeomorphism S (corresponding to b) that is constructed by taking some clopen castle decomposition with respect to T and doing some shuffling of the levels within each tower. This shuffling will be implemented by a permutation of the tower levels that sends the top level to the bottom one, which permits us to define S on the top level of each tower not according to the permutation (as on the other levels) but rather so as to be equal to T. As a consequence S will have the same orbits and the same asymptotic behaviour as T (in fact it will be conjugate to T). We thus formulate the following definition.

Definition 12.1. Let $T \curvearrowright X$ be a minimal homeomorphism of the Cantor set. A *tower permutation* of T is a homeomorphism $S \curvearrowright X$ for which there exists a clopen castle $\{(B_i, n_i)\}_{i=1}^k$ for T partitioning X and for each $i = 1, \dots k$ a cyclic permutation π_i of $\{0, \dots, n_i - 1\}$ with $\pi_i(n_i - 1) = 0$ such that $S = T^{\pi_i(j)-j}$ on $T^j B_i$ for $j = 0, \dots, n_i - 2$ and $S = T$ on $T^{n_i-1}B_i$.

If S is a tower permutation of T, then T is a tower permutation of S. Indeed, letting $i \in \{1, \dots, k\}$ we observe that $\pi_i^{n_i-1}(0) = n_i - 1$ and that S^j is equal to $T^{\pi_i^j(0)}$ on B_i for every $j \in \{0, \dots, n_i - 1\}$. In particular, $X = \bigsqcup_{i=1}^k \bigsqcup_{j=0}^{n_i-1} S^j B_i$ is a clopen castle decomposition for S. For every $j = 0, \dots, n_i - 2$, let $\sigma_i(j)$ be the unique integer in $\{1, \dots, n_i - 1\}$ satisfying the equation $\pi_i^j(0) + 1 = \pi_i^{\sigma_i(j)}(0)$, and set $\sigma_i(n_i - 1) = 0$. It is then straightforward to check that σ_i is a cyclic permutation of $\{0, \dots, n_i - 1\}$ such that $T = S^{\sigma_i(j)-j}$ on $S^j B_i$ and $T = S$ on $S^{n_i-1}B_i$.

This discussion in particular implies that if S is a tower permutation of T, then $\mathrm{orb}_T(x) = \mathrm{orb}_S(x)$ for all $x \in X$ and $M_T(X) = M_S(X)$.

Lemma 12.2. *Let $T \curvearrowright X$ be a minimal homeomorphism of the Cantor set, and let $S \curvearrowright X$ be a tower permutation of T. Then S and T are conjugate.*

Proof. Obtain a castle $\{(B_i, n_i)\}_{i=1}^k$ and cyclic permutations π_i for $i = 1, \dots, k$ from the definition of tower permutation. Given $1 \le i \le k$, the permutation π_i, being cyclic, is conjugate to the cyclic shift $\rho(j) = j + 1 \pmod{n_i}$ in $\mathrm{Sym}(n_i)$, i.e., there exists a permutation $\sigma_i \in \mathrm{Sym}(n_i)$ such that $\sigma_i \circ \pi_i = \rho \circ \sigma_i$. We can additionally ask that $\sigma_i(0) = 0$, which also forces $\sigma_i(n_i - 1) = n_i - 1$ by the nature of π and ρ, as one can easily check. Define $h : X \to X$ by

$$hx = T^{\sigma_i(j)-j}x$$

for all $x \in T^j B_i$, $i = 1, \dots, k$, and $j = 0, \dots, n_i - 1$. Since the tower levels form a clopen partition of X, this is a homeomorphism. Finally, we verify that h implements the desired conjugacy. Let $x \in X$. Then $x \in T^j B_i$ for some $1 \le i \le k$ and $0 \le j \le n_i - 1$. Suppose that $j \neq n_i - 1$. Then $Sx = T^{\pi_i(j)-j}x \in T^{\pi_i(j)}B_i$ and hence

$$hSx = T^{\sigma_i(\pi_i(j))-\pi_i(j)}Sx = T^{\sigma_i(\pi_i(j))-\pi_i(j)}T^{\pi_i(j)-j}x = T^{\sigma_i(\pi_i(j))-j}x$$

so that, using the equality $\sigma_i \circ \pi_i = \rho \circ \sigma_i$,

$$Thx = T^{\sigma_i(j)+1-j}x = T^{\sigma_i(\pi_i(j))-j}x = hSx.$$

Suppose finally that $j = n_i - 1$. Then on $T^j B_i$ we have $h = \mathrm{id}$ and $S = T$, and since h is also the identity on $B_{i'}$ for all $i' = 1, \dots, k$ and $T^{n_i}B_i \subseteq \bigsqcup_{i'=1}^k B_{i'}$ we obtain $Thx = Tx = hTx = hSx$ for every $x \in T^j B_i$. Therefore S and T are conjugate. □

If S is a tower permutation of a minimal homeomorphism T of the Cantor set X, then the resulting action $F_2 = \langle a, b \rangle \curvearrowright X$ defined by $a \mapsto T$ and $b \mapsto S$ satisfies $M_T(X) = M_S(X) = M_{F_2}(X)$. However, this action will in general be far from faithful, let alone topologically free. In order to remedy this we will construct diagonal products from such actions, in the same spirit as

Section 7, only now with infinitely many factors, which will permit us to arrange for topological freeness asymptotically via measure-theoretic considerations.

A family of p.m.p. actions $G \curvearrowright (Z_i, \zeta_i)$ with countable index set I is *disjoint* if the only probability measure on $\prod_{i\in I} Z_i$ that is invariant under the diagonal action of G and projects factorwise onto ζ_i for every $i \in I$ is the product measure $\prod_{i\in I} \zeta_i$. A family of p.m.p. actions $G \curvearrowright (Z_i, \zeta_i)$ with arbitrary index set I is *disjoint* if every finite subfamily is disjoint. It is clear that these two definitions agree in the overlapping case when I is a countably infinite set.

Lemma 12.3. *Suppose G is amenable. Let $G \curvearrowright X_n$ for $n \in \mathbb{N}$ be strictly ergodic actions on compact metrizable spaces such that, writing μ_n for the unique measure in $M_G(X_n)$, the family of p.m.p. actions $G \curvearrowright (X_n, \mu_n)$ for $n \in \mathbb{N}$ is disjoint. Then the diagonal action $G \curvearrowright \prod_{n\in\mathbb{N}} X_n$ given by $s(x_n)_{n\in\mathbb{N}} = (sx_n)_{n\in\mathbb{N}}$ is strictly ergodic.*

Proof. Set $\mu = \prod_{n\in\mathbb{N}} \mu_n$. By the definition of disjointness, the action $G \curvearrowright \prod_{n\in\mathbb{N}} X_n$ has a unique invariant Borel probability measure, namely μ. Now for every $m \in \mathbb{N}$ the diagonal action $G \curvearrowright \prod_{n=1}^m X_n$ must be minimal, for any nonempty proper G-invariant subset for this action would support a G-invariant Borel probability measure by the amenability of G, and this measure would differ from the product measure $\prod_{n=1}^m \mu_n$ since the latter has full support given that each μ_n has full support by the minimality of $G \curvearrowright X_n$, leading us to a contradiction with the disjointness of the family $G \curvearrowright (X_n, \mu_n)$ for $n = 1, \dots, m$. It follows that if A is a nonempty closed G-invariant subset of $\prod_{n\in\mathbb{N}} X_n$ then for every $m \in \mathbb{N}$ the projection of A onto $\prod_{n=1}^m X_n$ is a nonempty closed G-invariant set and hence equal to $\prod_{n=1}^m X_n$, from which we deduce that $A = \prod_{n\in\mathbb{N}} X_n$. Thus $G \curvearrowright \prod_{n\in\mathbb{N}} X_n$ is minimal. □

Lemma 12.4. *Let E be a finite subset of $F_2 \setminus \{e\}$ and let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for every finite set X of cardinality at least N there exists an action $F_2 \curvearrowright X$ in which each of the two standard generators a and b acts transitively and for which*

$$|\{x \in X : sx = x\}| \leq \varepsilon |X|$$

for every $s \in E$.

Proof. Take a $q \in \mathbb{N}$ such that each element of E can be written as a word of length at most q in the generating set $D := \{a, a^{-1}, b, b^{-1}\}$ for F_2.

Let $J \geq 2$ be an integer satisfying $(4/J)|D^q| \leq \varepsilon/2$. It is well known that F_2 is residually finite, and so there exists a finite quotient $\pi : F_2 \to H$ such that none of the images $\pi(s)$ for $s \in E$ and $\pi(a^j)$ and $\pi(b^j)$ for $j = 1, \dots, J$ are equal to e, and in particular $|H| \geq J$. Let N be any integer larger than $4|H|/\varepsilon$ and let us show that this fulfils the requirements of the lemma.

Let n be any integer with $n \geq N$. Then there is a largest integer $r \geq 1$ satisfying $(r+1)|H| \leq n$. Note that by the choice of n we have

$$n - r|H| < \frac{\varepsilon n}{2}.$$

Set $H^+ := H \times \{1, \dots, r\}$ and enlarge this to some set H^{++} of cardinality n. Consider the diagonal action ρ of F_2 on H^+ coming from the left translation action $(s, h) \mapsto \pi(s)h$ on the first factor and the trivial action on the second, and extend this to an action of F_2 on H^{++}, which we also call ρ, by having each of a and b act in some arbitrary transitive way on $H^{++} \setminus H^+$. The transitive pieces $Y_1, \dots, Y_m$ into which H^{++} partitions under the action of ρ_a each have cardinality at least J by our assumptions on π from the first paragraph together with the fact that $|H^{++} \setminus H^+| \geq |H| \geq J$. Therefore $m \leq n/J$. For each $k = 1, \dots, m$ choose a $y_{k,1} \in Y_k$. Then Y_k has the form

$$\{y_{k,1}, \rho_a(y_{k,1}), \rho_{a^2}(y_{k,1}), \dots, \rho_{a^{j_k}}(y_{k,1})\}$$

for some $j_k \geq J \geq 2$. Set $y_{k,2} = \rho_{a^{j_k}}(y_{k,1})$. Similarly, the transitive pieces $Z_1, \dots, Z_{m'}$ into which the action of ρ_b partitions H^{++} each have cardinality at least J, so that $m' \leq n/J$, and we choose for each $k = 1, \dots, m'$ a $z_{k,1} \in Z_k$ and write $z_{k,2} \in Z_k$ for the image of $z_{k,1}$ under $\rho_{b^{l_k}}$, where $l_k \geq J \geq 2$ is the largest integer satisfying $\rho_{b^{l_k}}(z_{k,1}) \neq z_{k,1}$. Define a new action κ of F_2 on H^{++} by setting $\kappa_a(y_{k,2}) = y_{k+1,1}$ for $k = 1, \dots, m-1$ and $\kappa_a(y_{m,2}) = y_{1,1}$ and setting $\kappa_a(y) = \rho_a y$ for other $y \in H^{++}$, and likewise setting $\kappa_b(z_{k,2}) = z_{k+1,1}$ for $k = 1, \dots, m'-1$ and $\kappa_b(z_{m',2}) = z_{1,1}$ and setting $\kappa_b(y) = \rho_b y$ for other $y \in H^{++}$. Then both κ_a and κ_b act transitively.

For each $s \in E$ write $s = s_1 \cdots s_r$ with $s_1, \dots, s_r \in D$ and $1 \leq r \leq q$. Suppose that $x \in H^{++}$ and that none of the elements $x, \kappa_{s_r}(x), \kappa_{s_{r-1}s_r}(x), \dots, \kappa_s(x)$ is equal to $y_{k,1}$, $y_{k,2}$, $z_{k',1}$, or $z_{k',2}$ for $k = 1, \dots, m$ and $k' = 1, \dots, m'$. By the way κ is defined we then have $\kappa_s(x) = \rho_s(x)$. In particular, if $x \in H^+$ then $\kappa_s(x) \neq x$ since $\pi(s) \neq e$.

Since $E \subseteq D^q$ and D^q is symmetric, we conclude that the set of all points $x \in H^{++}$ for which $\kappa_s x = x$ is contained in the set

$$H^{++} \setminus H^+ \cup \bigcup_{t \in D^q} \kappa_t(\{y_{k,1}, y_{k,2} : k = 1, \dots, m\} \cup \{z_{k,1}, z_{k,2} : k = 1, \dots, m'\})$$

and hence

$$\begin{aligned} |\{x \in H^{++} : \kappa_s x = x\}| &\leq |H^{++} \setminus H| + 2(m + m')|D^q| \\ &\leq (n - r|H|) + \frac{4n}{J}|D^q| \leq \varepsilon n. \end{aligned}$$

Finally, let X be any set of cardinality n and let $\theta : X \to H^{++}$ be any bijection. Then the action of F_2 on X defined by $s \mapsto \theta^{-1}\kappa_s\theta$ for $s \in F_2$ has the desired properties. □

Lemma 12.5. *Let $T \curvearrowright X$ be a minimal homeomorphism of the Cantor set. Let E be a finite subset of $F_2 \setminus \{e\}$ and let $\varepsilon > 0$. Then there exists a tower permutation S of T such that, for the induced action $F_2 = \langle a, b \rangle \curvearrowright X$ sending $a \mapsto T$ and $b \mapsto S$, one has $\mu(\{x \in X : sx = x\}) \leq \varepsilon$ for every $s \in E$ and $\mu \in M_T(X)$.*

Proof. Take a $q \in \mathbb{N}$ such that each element of E can be written as a word of length at most q in the generating set $D := \{a, a^{-1}, b, b^{-1}\}$ for F_2.

As recalled at the beginning of the section, by minimality T admits a clopen castle $\{(B_i, n_i)\}_{i=1}^k$ partitioning X such that each $n_i \geq 2$ is an integer greater than $4|D^q|/\varepsilon$ that is also sufficiently large for a purpose to be described imminently.

Let $1 \leq i \leq k$. Then by Lemma 12.4 we can assume n_i to be large enough so that there exists an action $F_2 \overset{\kappa}{\curvearrowright} \{1, \dots, n_i - 2\}$ such that κ_a is the cyclic permutation sending j to $j+1$ for $j = 1, \dots, n_i - 3$ and sending $n_i - 2$ to 1, the action of κ_b is transitive, and

$$|\{x \in \{1, \dots, n_i - 2\} : \kappa_s x = x\}| \leq \frac{\varepsilon}{2}(n_i - 2) \tag{12.1}$$

for every $s \in E$. Define S on the ith tower by setting

(i) $S = T^{\kappa_b(1)}$ on B_i,
(ii) $S = T^{n_i - 2}$ on TB_i,

(iii) $S = T^{\kappa_b(l)-l}$ on $T^l B_i$ for $2 \leq l \leq n_i - 2$,
(iv) $S = T$ on $T^{n_i-1}B_i$.

Having done this over all $i = 1, \dots, k$, we obtain a tower permutation S of T via the cyclic permutations $\pi_i = (0, \kappa_b(1), \dots, \kappa_b^{n_i-3}(1), 1, n_i-1) \in \mathrm{Sym}(n_i)$. Define an action $F_2 = \langle a, b\rangle \overset{\alpha}{\curvearrowright} X$ by setting $\alpha_a = T$ and $\alpha_b = S$.

Now let $s \in E$ and set $X_s = \{x \in X : \alpha_s x = x\}$. Let $\mu \in M_T(X)$. Given (12.1) and the way that we constructed S on each tower, we see that the union of the levels in the ith tower that do not get sent under α_s to a different level in the same tower has μ-measure at most $(\varepsilon/2)(n_i - 2)\mu(B_i) + 2|D^q|\mu(B_i)$, with the second summand accounting for the possibility that, under α, the successive application of the generators in D that spell out a reduced word in E sends us at some point into the bottom or top level of the tower. More explicitly, write $C_i = \bigsqcup_{l=0}^{n_i-1} T^l B_i$ and let $x \in C_i$. If $\alpha_s x \notin C_i$, then x must belong to the union

$$\bigcup_{t\in D^q} \alpha_t(B_i \cup T^{n_i-1}B_i),$$

which has μ-measure bounded by $2|D^q|\mu(B_i)$ given that $M_T(X) = M_S(X)$. If x is not in that union then $x \in \bigsqcup_{l=1}^{n_i-2} T^l B_i$ and one can show by a straightforward induction on the word length of s that $\alpha_s x \in T^{\kappa_s(l)}B_i$ whenever $x \in T^l B_i$. Therefore

$$X_s \cap C_i \subseteq \left(\bigcup_{t\in D^q} \alpha_t(B_i \cup T^{n_i-1}B_i)\right) \cup \left(\bigsqcup_{l=1}^{n_i-2} T^l B_i \cap X_s\right)$$

and so, using (12.1) and the fact that $n_i > 4|D^q|/\varepsilon$,

$$\mu(X_s \cap C_i) \leq 2|D^q|\mu(B_i) + \frac{\varepsilon}{2}(n_i - 2)\mu(B_i) \leq \varepsilon n_i \mu(B_i) = \varepsilon\mu(C_i).$$

This yields finally

$$\mu(X_s) = \sum_{i=1}^{k} \mu(X_s \cap C_i) \leq \varepsilon \sum_{i=1}^{k} \mu(C_i) = \varepsilon,$$

giving the desired conclusion. □

We are now in a position to establish the first main result of this section, which produces actions in both $\mathrm{WA}(F_2, X)$ and $\mathrm{A}^*(F_2, X)$. We note that sequences $(T_n)_{n\in\mathbb{N}}$ as in the proposition statement exist in abundance, as explained in Remark 12.7.

Proposition 12.6. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of strictly ergodic homeomorphisms of the Cantor set X such that, writing μ_n for the unique measure in $M_{T_n}(X)$, the p.m.p. transformations $T_n \curvearrowright (X, \mu_n)$ for $n \in \mathbb{N}$ are weakly mixing and form a disjoint family. Then there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of homeomorphisms of X such that, defining for each n the action $F_2 = \langle a, b\rangle \overset{\alpha_n}{\curvearrowright} X$ by $a \mapsto T_n$ and $b \mapsto S_n$, one has the following:*

(i) $M_{\alpha_n}(X) = M_{T_n}(X) = M_{S_n}(X)$ *for every* $n \in \mathbb{N}$,
(ii) $\lim_{n\to\infty} \mu_n(\{x \in X : \alpha_{n,s}x = x\}) = 0$ *for every* $s \in F_2 \setminus \{e\}$,
(iii) the diagonal action $F_2 \curvearrowright X^{\mathbb{N}}$ *is topologically free, topologically weakly mixing, and strictly ergodic,*

(iv) the homeomorphisms $T = \prod_{n\in\mathbb{N}} T_n$ and $S = \prod_{n\in\mathbb{N}} S_n$ of $X^{\mathbb{N}}$ are strictly ergodic and topologically weakly mixing.

In particular, any conjugate $F_2 \curvearrowright X$ of the diagonal action $F_2 \curvearrowright X^{\mathbb{N}}$ by a homeomorphism $X^{\mathbb{N}} \to X$ belongs to both $\mathrm{WA}(F_2, X)$ *and* $\mathrm{A}^*(F_2, X)$.

Proof. Take an increasing sequence $K_1 \subseteq K_2 \subseteq \dots$ of finite subsets of $F_2 \setminus \{e\}$ with $\bigcup_{n=1}^{\infty} K_n = F_2 \setminus \{e\}$. For each n apply Lemma 12.5 to obtain a tower permutation S_n of T_n such that the action α_n of $F_2 = \langle a, b\rangle$ given by $a \mapsto T_n$ and $b \mapsto S_n$ satisfies $\mu_n(X_{\alpha_n,s}) \leq 1/n$ for all $s \in K_n$ where $X_{\alpha_n,s}$ denotes the fixed-point set $\{x \in X : \alpha_{n,s}x = x\}$. It follows that for every $s \in F_2\setminus\{e\}$ we have $\lim_{n\to\infty} \mu_n(X_{\alpha_n,s}) = 0$. As noted earlier, it follows readily from the definition of tower permutation that $M_{S_n}(X) = M_{T_n}(X) = M_{\alpha_n}(X) = \{\mu_n\}$ for all n.

It is now easy to see that the actions $F_2 \overset{\alpha_n}{\curvearrowright} X$ are strictly ergodic and the family of p.m.p. actions $F_2 \overset{\alpha_n}{\curvearrowright} (X, \mu_n)$ for $n \in \mathbb{N}$ is disjoint. By Lemma 12.3, the diagonal action $\prod_{n\in\mathbb{N}} \alpha_n$ of F_2 on $X^{\mathbb{N}}$, which is determined by $a \mapsto T = \prod_{n\in\mathbb{N}} T_n$ and $b \mapsto S = \prod_{n\in\mathbb{N}} S_n$, is strictly ergodic, and its unique invariant measure is $\mu = \prod_{n\in\mathbb{N}} \mu_n$.

Again applying Lemma 12.3, the action $T \curvearrowright X^{\mathbb{N}}$ is strictly ergodic, and as a p.m.p. transformation of $(X^{\mathbb{N}}, \mu)$ it is weakly mixing since each factor is weakly mixing. Since by minimality each μ_n has full support, so does μ. A simple exercise then shows that T is topologically weakly mixing, and therefore so is the diagonal action $F_2 \curvearrowright X^{\mathbb{N}}$.

By Lemma 12.2 the homeomorphism S is conjugate to T and hence is strictly ergodic and topologically weakly mixing.

It remains to show that the action $F_2 \curvearrowright X^{\mathbb{N}}$ is topologically free. For every $s \in F_2 \setminus \{e\}$ we see, given the equality $\{x \in X^{\mathbb{N}} : sx = x\} = \prod_{n\in\mathbb{N}} X_{\alpha_n,s}$, that

$$\mu(\{x \in X^{\mathbb{N}} : sx = x\}) = \lim_{n\to\infty} \prod_{k=1}^{n} \mu_k(X_{\alpha_k,s}) \leq \lim_{n\to\infty} \mu_n(X_{\alpha_n,s}) = 0,$$

which is tantamount to saying that the p.m.p. action $F_2 \curvearrowright (X^{\mathbb{N}}, \mu)$ is free. From this we deduce that the action $F_2 \curvearrowright X^{\mathbb{N}}$, being minimal, must be topologically free. □

Remark 12.7. Examples of sequences $(T_n)_{n\in\mathbb{N}}$ as in Proposition 12.6 can be produced as follows. Del Junco showed that there is an uncountable disjoint family $\{T_i\}_{i\in I}$ of weakly mixing p.m.p. transformations [17, Corollary 2(b)], and by the Jewett–Krieger theorem we may view each T_i as a minimal homeomorphism of the Cantor set with the invariant Borel probability measure being unique. Then any sequence drawn from this family will fulfill the requirements.

Our remaining task is to establish square divisibility for the actions appearing in Proposition 12.6. As observed in Remark 3.11, this will also imply weak square divisibility. We reprise, in a simplified form, some of the arguments at play in Sections 9 and 10. The first of the lemmas we will need is naturally phrased in terms of topological full groups.

Definition 12.8. Let $G \curvearrowright X$ be a minimal action on the Cantor set. The *topological full group*, denoted $[[G \curvearrowright X]]$, is the group of all homeomorphisms $h : X \to X$ such that there exists a finite clopen partition $\{A_1, \dots, A_n\}$ of X and $s_1, \dots, s_n \in G$ such that $hx = s_i x$ for all $i = 1, \dots, n$ and $x \in A_i$. In the case of a single homeomorphism $T \curvearrowright X$ we write $[[T]]$ for the topological full group of the action of $\mathbb{Z}$ that it generates. Note that when the action $G \curvearrowright X$ is faithful one has a canonical embedding of G into $[[G \curvearrowright X]]$.

Lemma 12.9. *Let $G \curvearrowright X$ be a faithful minimal action on the Cantor set and let $O_1, O_2 \subseteq X$ be nonempty clopen subsets. Suppose that for every finite set $e \in E \subseteq G$ the action is (O_1, O_2, E)-squarely divisible. Then for every subgroup $H \subseteq [[G \curvearrowright X]]$ satisfying $G \subseteq H$ (identifying G with its image under the canonical embedding into $[[G \curvearrowright X]]$ that we get from faithfulness) and every finite set $e \in F \subseteq H$, the action $H \curvearrowright X$ associated with the inclusion of H in the homeomorphism group of X is (O_1, O_2, F)-squarely divisible.*

Proof. Let $e \in F \subseteq H$ be a finite set. By assumption, for each $h \in F$ there exists a clopen partition $\{A_{h,1}, \dots, A_{h,n_h}\}$ of X and elements $s_{h,i} \in G$ such that $hx = s_{h,i}x$ for all $i = 1, \dots, n_h$ and $x \in A_{h,i}$. Write the join of these partitions as $\{B_1, \dots, B_m\}$. This is a clopen partition with the property that for every $h \in F$ and $j = 1, \dots, m$ there exists an $s_{h,j} \in G$ such that $hx = s_{h,j}x$ for all $x \in B_j$, as $B_j \subseteq A_{h,i}$ for some i. Write E for the finite subset $\{s_{h,j} : h \in F,\ 1 \leq j \leq m\}$ of G. By hypothesis, the action $G \curvearrowright X$ is (O_1, O_2, E)-squarely divisible. Hence by Proposition 3.9 there exist an $n \in \mathbb{N}$ and pairwise equivalent and E-disjoint clopen subsets $\{V_{p,q}\}_{p,q=1}^n$ of X such that, writing $V = \bigsqcup_{p,q=1}^n V_{p,q}$, $V_1 = \bigsqcup_{p=1}^n V_{p,1}$, $R = V^c$, and $B = V \cap (V^E)^c$, the following hold:

(i) $V_{p,1} \prec O_1 \cap \bigsqcup_{q=2}^n V_{p,q} \cap B^c$ for every $p = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V_1 \cup B)^c$,
(iii) $B \prec O_2 \cap R$.

Now from the definition of E we see that $FY \subseteq EY$ and $Y^F \supseteq Y^E$ for any set $Y \subseteq X$. The first of these inclusions implies that the sets $\{V_{p,q}\}_{p,q=1}^n$ are F-disjoint for the action of H. The second implies that the set $\tilde{B} := V \cap (V^F)^c$ is contained in $V \cap (V^E)^c$, and so from (i)–(iii) we immediately obtain the following subequivalences with respect to the action of G and hence also with respect to the action of H (since $G \subseteq H \subseteq [[G \curvearrowright X]]$ by hypothesis):

(i) $V_{p,1} \prec O_1 \cap \bigsqcup_{q=2}^n V_{p,q} \cap \tilde{B}^c$ for every $p = 1, \dots, n$,
(ii) $R \prec O_2 \cap V \cap (V \cup \tilde{B})^c$,
(iii) $\tilde{B} \prec O_2 \cap R$.

Notice furthermore that the sets $\{V_{p,q}\}_{p,q=1}^n$ are pairwise equivalent for the action of H since they are pairwise equivalent for the action of G. We conclude by Proposition 3.9 that $H \curvearrowright X$ is (O_1, O_2, F)-squarely divisible. □

Lemma 12.10. *Let $T \curvearrowright X$ be a minimal homeomorphism of the Cantor set and S a tower permutation of T. Let $F_2 = \langle a, b \rangle \curvearrowright X$ be the action given via $a \mapsto T$ and $b \mapsto S$. Let $O_1, O_2 \subseteq X$ be nonempty clopen sets and let $e \in E \subseteq F_2$ be a finite set. Then the action $F_2 \curvearrowright X$ is (O_1, O_2, E)-squarely divisible.*

Proof. Since $T \curvearrowright X$ is almost finite (as is manifest by the clopen castle decompositions discussed at the beginning of this section and the previous one), by Theorem 5.7 the action $\langle a \rangle \curvearrowright X$ is (O_1, O_2, F)-squarely divisible for every finite set $e \in F \subseteq \langle a \rangle$. Also, the minimality of T and the infiniteness of X together imply that the action $\langle a \rangle \curvearrowright X$ is free (and hence faithful). Since it is clear from the definition of tower permutation that $S \in [[T]]$, we deduce by Lemma 12.9 that the action $F_2 \curvearrowright X$ given by $a \mapsto T$ and $b \mapsto S$ is (O_1, O_2, E)-squarely divisible. □

Remark 12.11. In Lemma 12.10 the action $F_2 \curvearrowright X$ is not faithful. In fact a theorem of Juschenko and Monod [42] implies that the action $F_2 \curvearrowright X$ must factor through an action of an amenable group.

As Lemma 12.10 illustrates, Lemma 12.9 yields many examples of squarely divisible actions of (frequently nonamenable) groups on the Cantor set. However, as Remark 12.11 highlights, these actions are often far from being topologically free or even faithful. We now recapitulate a method that we have already used on a couple of occasions for showing square divisibility in the context diagonal actions (cf. Lemma 9.1).

Lemma 12.12. *Let $G \overset{\alpha_n}{\curvearrowright} X_n$ for $n \in \mathbb{N}$ be actions on the Cantor set such that the corresponding diagonal action $G \overset{\alpha}{\curvearrowright} X := \prod_{n\in\mathbb{N}} X_n$ is minimal. Suppose that each α_n is (O_1, O_2, E)-squarely divisible for all nonempty clopen sets $O_1, O_2 \subseteq X_n$ and finite sets $e \in E \subseteq G$. Suppose that for every finite subset $F \subseteq G$ and $M \in \mathbb{N}$ there exist an integer $m \geq M$ and a nonempty clopen set $A \subseteq X_m$ such that (F, A) is a tower for α_m. Then the action $G \curvearrowright X$ is squarely divisible.*

Proof. Let O be a nonempty clopen subset of X. By Proposition 3.9 it is enough to establish O-square divisibility, and for this we may assume, by passing to a smaller clopen set if necessary, that O satisfies $O = \pi_I^{-1}(\pi_I(O))$ for some finite set $I \subseteq \mathbb{N}$ where $\pi_I : X \to X_I := \prod_{n\in I} X_n$ is the coordinate projection map. Since α is minimal so is $\gamma := \prod_{n\in I} \alpha_n$, and so we can apply a compactness argument as in the proof of Lemma 9.1 to find a finite set $F \subseteq G \setminus \{e\}$ and a clopen partition $\{W_s\}_{s\in F}$ of X_I indexed by F such that $\gamma_s W_s \subseteq \pi_I(O)$ for each $s \in F$.

Let $E \subseteq G$ be a finite set containing e. By hypothesis there are an $m > \max I$ and a clopen set $A \subseteq X_m$ such that $(F \cup \{e\}, A)$ is a tower for α_m. Take disjoint nonempty clopen sets $O_1, O_2 \subseteq A$. Since $G \overset{\alpha_m}{\curvearrowright} X_m$ is (O_1, O_2, E)-squarely divisible, by Proposition 3.9 there exist an $n \in \mathbb{N}$ and pairwise equivalent and pairwise E-disjoint clopen subsets $\{V_{i,j}\}_{i,j=1}^n$ of X_m such that, writing $V = \bigsqcup_{i,j=1}^n V_{i,j}$, $V_1 = \bigsqcup_{i=1}^n V_{i,1}$, $R = X_m \setminus V$, and $B = V \cap (V^E)^c$,

(i) $V_{i,1} \prec_{\alpha_m} O_1 \cap \bigsqcup_{j=2}^n V_{i,j} \cap B^c$ for $i = 1, \dots, n$,
(ii) $R \prec_{\alpha_m} O_2 \cap V \cap (V_1 \cup B)^c$,
(iii) $B \prec_{\alpha_m} O_2 \cap R$.

For each $C \subseteq X_m$ set $C' = \prod_{n=1}^{m-1} X_n \times C \times \prod_{n=m+1}^{\infty} X_n \subseteq \prod_{n\in\mathbb{N}} X_n$. Then from the subequivalences above we get

(i) $V'_{i,1} \prec_\alpha O'_1 \cap \bigsqcup_{j=2}^n V'_{i,j} \cap (B')^c$ for $i = 1, \dots, n$,
(ii) $R' \prec_\alpha O'_2 \cap V' \cap (V'_1 \cup B')^c$,
(iii) $B' \prec_\alpha O'_2 \cap R'$.

The properties of E-disjointness and pairwise equivalence of the clopen sets $V'_{i,j}$ are inherited from the $V_{i,j}$. Hence the action $G \overset{\alpha}{\curvearrowright} X$ is (O'_1, O'_2, E)-squarely divisible. Write π for the coordinate projection map $\prod_{n\in\mathbb{N}\setminus\{m\}} X_n \to X_I$. Since $\gamma_s W_s \subseteq \pi_I(O)$ for $s \in F$ and the sets $\alpha_{m,s}(O_1 \sqcup O_2)$ for $s \in F$ are pairwise disjoint, we obtain, interpreting the product sets below via the identification of X with $(\prod_{n\in\mathbb{N}\setminus\{m\}} X_n) \times X_m$,

$$
\begin{aligned}
O'_1 \sqcup O'_2 = \bigsqcup_{s\in F} \pi^{-1}(W_s) \times (O_1 \sqcup O_2) &\sim \bigsqcup_{s\in F} s(\pi^{-1}(W_s) \times (O_1 \sqcup O_2)) \\
&\subseteq \pi^{-1}(\pi_I(O)) \times (X_m \setminus (O_1 \sqcup O_2)).
\end{aligned}
$$

This shows that $O'_1 \sqcup O'_2 \prec \pi^{-1}(\pi_I(O)) \times (X_m \setminus (O_1 \sqcup O_2))$, and since the two sets in this subequivalence are disjoint and the second one is contained in O, we conclude by Proposition 3.9 that $G \curvearrowright X$ is O-squarely divisible. □

Proposition 12.13. *Let $F_2 \curvearrowright X$ be an action on the Cantor set as in the last sentence of Proposition 12.6. Then $F_2 \curvearrowright X$ is squarely divisible.*

Proof. By assumption there are T_n, S_n, and μ_n as in Proposition 12.6 so that, up to conjugacy, we can decompose the action $F_2 \curvearrowright X$ as a diagonal product $F_2 = \langle a, b\rangle \curvearrowright X^{\mathbb{N}}$ of actions $F_2 \overset{\alpha_n}{\curvearrowright} X$ generated by $a \mapsto T_n$ and $b \mapsto S_n$ with

$$\lim_{n\to\infty} \mu_n(\{x \in X : \alpha_{n,s}x = x\}) = 0 \tag{12.2}$$

for every $s \in F_2\backslash\{e\}$. By Lemma 12.10 each α_n is (O_1, O_2, E)-squarely divisible for all nonempty clopen sets $O_1, O_2 \subseteq X$ and finite sets $e \in E \subseteq F_2$.

Now given a finite set $F \subseteq F_2$ and an $M \in \mathbb{N}$, by (12.2) we can find an integer $m \geq M$ such that $\mu_m(\{x \in X : \alpha_{m,s}x = x\}) \leq 1/(2|F^{-1}F|)$ for all $s \in F^{-1}F \setminus \{e\}$, which implies that the set of all $x \in X$ such that $\alpha_{m,s}x \neq x$ for all $s \in F^{-1}F \setminus \{e\}$ has μ_m-measure at least $1/2$. Choosing a particular x in this set we then have $\alpha_{m,s}x \neq \alpha_{m,t}x$ for all distinct $s, t \in F$, which permits us to find clopen neighbourhood $A \subseteq X$ of x such that (F, A) is a tower for α_m. We have thus verified the conditions that enable us to apply Lemma 12.12 and hence conclude that $F_2 \curvearrowright X$ is squarely divisible. □

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